① ~~$\mathbb{R}^+$~~

Espacio $\rightarrow \mathbb{R}^+$

Op interna $\rightarrow *$, definida como $x * y = x \cdot y$, $x, y \in \mathbb{R}^+$

Op. externa $\rightarrow \circ$, definida como $\lambda \circ x = x^{\lambda}$, $x \in \mathbb{R}^+$, $\lambda \in \mathbb{K}\ (\mathbb{R})$

Demostrar que $(\mathbb{R}, *, \circ)$ es ev.

props. de grupo abeliano:

i) Asociativa de $*$: Sean $x, y, z \in \mathbb{R}^+$, $x * (y * z) = x \cdot (y \cdot z) = (x \cdot y) \cdot z = (x * y) * z$ ■

ii) Elemento neutro $*$: Sea $x \in \mathbb{R}^+$, $\exists 1 \in \mathbb{R}^+ \mid x * 1 = x \cdot 1 = x$ ■

iii) Elemento inverso $*$: Sea $x \in \mathbb{R}^+$, $\exists x^{-1} \in \mathbb{R}^+ \mid x * x^{-1} = x \cdot x^{-1} = 1$ ■

iv) Conmutativa $*$: Sean $x, y \in \mathbb{R}^+$, $x * y = x \cdot y = y \cdot x = y * x$ ■

v) Asociativa $\circ$: Sea $x \in \mathbb{R}^+$, $\lambda, \mu \in \mathbb{R}$, $\lambda \circ (\mu \circ x) = (x^{\mu})^{\lambda} = x^{\mu\lambda} = (\lambda \cdot \mu) \circ x$ ■

vi) Elemento neutro $\circ$: Sea $x \in \mathbb{R}^+$, $\exists 1 \in \mathbb{R} \mid 1 \circ x = x^1 = x$ ■

vii) Distributiva $\circ \sim *$: Sean $x, y \in \mathbb{R}^+$, $\lambda \in \mathbb{R}$, $\lambda \circ (x * y) = (x \cdot y)^{\lambda} = x^{\lambda} \cdot y^{\lambda} = (\lambda \circ x) * (\lambda \circ y)$ ■

viii) Distributiva $\circ \sim +$: Sea $x \in \mathbb{R}^+$, $\lambda, \mu \in \mathbb{R}$, $(\lambda + \mu) \circ x = x^{(\lambda+\mu)} = x^{\lambda} \cdot x^{\mu} = (\lambda \circ x) * (\mu \circ x)$ ■

EJERCICIOS DE ESPACIOS VECTORIALES:

1) Sea $\mathbb{R}^+$ con operación interna "producto" $x * y = x \cdot y$ y sea $\mathbb{R}$, operación externa $\lambda \circ x = x^\lambda$, $\forall \lambda \in \mathbb{R}, \forall x \in \mathbb{R}^+$. Demuestra que $(\mathbb{R}^+, *, \circ)$ es ev.

Será ev. si cumple las ocho propiedades de ev.:

props. de grupo abeliano

i) Asociativa de $*$: Sean $x, y, z \in \mathbb{R}^+$, $x * (y * z) = x \cdot (y \cdot z) = (x \cdot y) \cdot z = (x * y) * z$ ■

ii) Elemento neutro $*$: Sea $x \in \mathbb{R}^+$, $x * 0_E = x \Rightarrow 0_E = 1$ y $1 \in \mathbb{R}^+ \Rightarrow \exists\, 0_E \in \mathbb{R}^+$ ■

iii) Elemento inverso $*$: Sea $x \in \mathbb{R}$, $x * x_I = 0_E \Rightarrow x \cdot x^{-1} = 1$ y $x^{-1} = \frac{1}{x} \in \mathbb{R}^+$, ya que $x \in \mathbb{R}^+$, luego $\exists\, x_I$ elemento inverso $*$ en $\mathbb{R}^+$ ■

iv) Conmutativa $*$: Sea $x, y \in \mathbb{R}^+$, $x * y = x \cdot y = y \cdot x = y * x$ ■

cuando trabajamos con op. int/ext. de escalares seguimos usando $(+, \cdot)$ convenientes

v) Asociativa $\circ$: Sean $x \in \mathbb{R}^+$, $\lambda, \mu \in \mathbb{R}$, $\lambda \circ (\mu \circ x) = (x^\mu)^\lambda \neq x^{(\mu^\lambda)} = (\lambda \circ \mu) \circ x$

NO CUMPLE ESTA PROPIEDAD, luego $(\mathbb{R}^+, *, \circ)$ no es ev.

$\lambda \cdot (\mu \circ x) \overset{?}{=} (\lambda \cdot \mu) \circ x$ ✓

MAL

2) ¿Cuáles son sub-ev? de $\mathbb{R}^3$, nótese que el $\bar{0}_E \in \mathbb{R}^3$, $\bar{0}_E = (0,0,0)$

a) $J_1 = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$

i) ¿$\bar{0}_E \in J_1$? Se verifica que $0 = 0^2 + 0^2$, luego $\bar{0}_E \in J_1$ ■

ii) ¿$\lambda \bar{x} + \bar{y} \in J_1$, $\forall \bar{x}, \bar{y} \in \mathbb{R}^3$? Sean $\bar{x} = (x_1, x_2, x_3)$, $\bar{y} = (y_1, y_2, y_3)$ tales que $x_3 = x_1^2 + x_2^2$ y $y_3 = y_1^2 + y_2^2$

$$\lambda \bar{x} + \bar{y} = (\lambda x_1 + y_1, \lambda x_2 + y_2, \lambda x_3 + y_3)$$

¿Verifica $(\lambda \bar{x} + \bar{y})$ $z = x^2 + y^2$? $\lambda x_3 + y_3 = (\lambda x_1 + y_1)^2 + (\lambda x_2 + y_2)^2$

Sean, por ejemplo, $\bar{x} = (1, 2, 5)$ y $\bar{y} = (3, 2, 13)$, $\lambda = 1 \Rightarrow 18 \overset{?}{=} (4)^2 + (4)^2$ ✗ no verifica, luego

J_1 no es subev. de $\mathbb{R}^3$

b) $J_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$

i) ¿$\bar{0}_E \in J_2$? Vemos que $\bar{0}_E = (0,0,0)$ verifica $x = 0$, luego $\bar{0}_E \in J_2$ ■

ii) Sean $\bar{x} = (0, x_2, x_3)$, $\bar{y} = (0, y_2, y_3) \in J_2$, $\forall \lambda \in \mathbb{R}$, vemos que el vector $(\lambda \bar{x} + \bar{y}) = (0, \lambda x_2 + y_2, \lambda x_3 + y_3)$ verifica $x = 0$, luego $\lambda \bar{x} + \bar{y} \in J_2$ y J_2 es subev. de $\mathbb{R}^3$

c) $J_3 = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - 3y + 5z = 0\}$

i) ¿$\bar{0}_E \in J_3$? $\bar{0}_E = (0,0,0)$ verifica $2 \cdot 0 - 3 \cdot 0 + 5 \cdot 0 = 0$ ✓, luego $\bar{0}_E \in J_3$

ii) Sean $\bar{x} = (x_1, x_2, x_3)$, $\bar{y} = (y_1, y_2, y_3) \in J_3$, $\forall \lambda \in \mathbb{R}$, ¿se verifica $\lambda \bar{x} + \bar{y} \in J_3$?

$$\lambda \bar{x} + \bar{y} = (\lambda x_1 + y_1, \lambda x_2 + y_2, \lambda x_3 + y_3)$$

$$2(\lambda x_1 + y_1) - 3(\lambda x_2 + y_2) + 5(\lambda x_3 + y_3) = 2\lambda x_1 + 2y_1 - 3\lambda x_2 + 3y_2 + 5\lambda x_3 + 5y_3 =$$

ya que $\bar{x}, \bar{y} \in J_3$ $\rightsquigarrow$ $\lambda \underbrace{(2x_1 - 3x_2 + 5x_3)}_{=0} + \underbrace{2y_1 + 3y_2 + 5y_3}_{=0} = \lambda \cdot 0 + 0 = 0$ ✓ ■ Luego J_3 es subev. de $\mathbb{R}^3$

d) $J_4 = \{(x,y,z) \in \mathbb{R}^3 / x - 2y - 3z = 1\}$

i) $¿\bar{0}_E \in J_4?$ $\quad \bar{0}_E = (0,0,0)$, $\quad 0 - 2\cdot 0 - 3\cdot 0 \neq 1$, no verifica $\Rightarrow$ $\underline{J_4 \text{ no es subev de } \mathbb{R}^3}$

e) $J_5 = \{(x,y,z) \in \mathbb{R}^3 / x = y = z = 0\}$ $\Leftrightarrow$ $J_5 = \{\bar{0}\}$, $\underline{\text{es sub ev. de } \mathbb{R}^3.}$

i) $\bar{0}_E \in J_5 \Leftarrow 0 = 0 = 0 = 0$ ■

ii) $\lambda\bar{x} + \bar{y} = (\lambda 0 + 0, \lambda 0 + 0, \lambda 0 + 0) = (0,0,0)$, $\quad 0 = 0 = 0 = 0 \Rightarrow \lambda\bar{x} + \bar{y} \in J_5$ ■

f) $J_6 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$

i) $\bar{0}_E \overset{?}{\in} J_6$; $\quad 0^2 + 0^2 + 0^2 \neq 1$, no verifica $\Rightarrow$ $\underline{J_6 \text{ no es subev. de } \mathbb{R}^3}$

g) $J_7 = \{(x,y,z) \in \mathbb{R}^3 / \overset{(i)^*}{x = 2y}, \overset{(ii)^*}{2x - 3y + 5z = 0}\}$

i) $\bar{0}_E \overset{?}{\in} J_7$; $\quad 0 = 2\cdot 0 \checkmark$, $\quad 2\cdot 0 - 3\cdot 0 + 5\cdot 0 = 0 \checkmark$ $\quad\Rightarrow \bar{0}_E \in J_7$ ■

ii) $\forall \bar{x}, \bar{y} \in J_7$, $\forall \lambda \in \mathbb{R}$, $¿\lambda\bar{x} + \bar{y} \in J_7?$, Sean $\bar{x} = (x_1, x_2, x_3)$ y $\bar{y} = (y_1, y_2, y_3) \in J_7$

$\lambda\bar{x} + \bar{y} = (\lambda x_1 + y_1, \lambda x_2 + y_2, \lambda x_3 + y_3)$

$(i)^*$ $\lambda x_1 + y_1 \overset{?}{=} 2(\lambda x_2 + y_2) \Leftrightarrow \lambda x_1 + y_1 \overset{?}{=} 2\lambda x_2 + 2y_2 \Leftrightarrow \lambda(x_1) + y_1 \overset{?}{=} \lambda(2x_2) + 2y_2$

Dado que $\bar{x}, \bar{y} \in J_7$, $\left.\begin{array}{l} \cancel{\lambda}(x_1) = \cancel{\lambda}(2x_2) \\ y_1 = 2y_2 \end{array}\right\} \Rightarrow \lambda(x_1) + y_1 \overset{\checkmark}{=} \lambda(2x_2) + 2y_2$ ■ (i)

$(ii)^*$ $2(\lambda x_1 + y_1) - 3(\lambda x_2 + y_2) + 5(\lambda x_3 + y_3) \overset{?}{=} 0$; $\quad 2\lambda x_1 + 2y_1 - 3\lambda x_2 - 3y_2 + 5\lambda x_3 + 5y_3 \overset{?}{=} 0$;

$\lambda(\underbrace{2x_1 - 3x_2 + 5x_3}_{0}) + \underbrace{2y_1 - 3y_2 + 5y_3}_{0} \overset{?}{=} 0 \Rightarrow \lambda\cdot 0 + 0 = 0 \checkmark$ ■ (ii)

$\underline{\text{luego } J_7 \text{ es subev. de } \mathbb{R}^3}$

h) $J_8 = \{(x,y,z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}$

i) $\bar{0}_E \overset{?}{\in} J_8$; $\quad 0 = 0^2 - 0^2 \checkmark \Rightarrow \bar{0}_E \in J_8$

ii) Sean $\bar{x} = (2,3,-5)$, $\bar{y} = (1,2,-3) \in J_8$, $\lambda = 2 \in \mathbb{R}$

$\lambda\bar{x} + \bar{y} = (5,8,-13)$; $\quad -13 \overset{?}{=} 5^2 - 8^2 = -39$ ✗ no verifica (ii), luego

$\underline{J_8 \text{ no es subev. de } \mathbb{R}^3}$

i) $S_9 = \{(x,y,z) \in \mathbb{R}^3 \mid z = x \cdot y\}$

i) $\overline{O}_E \overset{?}{\in} S_9$; $0 = 0 \cdot 0$ ✓ $\Rightarrow$ $\overline{O}_E \in S_9$ ■

ii) $\overline{x} = (1,2,2)$, $\overline{y} = (3,4,12) \in S_9$, $\lambda = 2$, $\Rightarrow$ $\lambda\overline{x} + \overline{y} = (5,8,16)$

¿ $\lambda\overline{x} + \overline{y} \overset{?}{\in} S_9$? ; $5 \cdot 8 = 40 \neq 16$ $\Rightarrow$ $\lambda\overline{x} + \overline{y} \notin S_9$ $\Rightarrow$ S_9 no es subesv. de $\mathbb{R}^3$

j) $S_{10} = \{(x,y,z) \in \mathbb{R}^3\} = \mathbb{R}^3$, $\mathbb{R}^3$ es subesv. de $\mathbb{R}^3$

i) $\overline{O}_E \in \mathbb{R}^3 \Leftrightarrow \overline{O}_E \in \mathbb{R}^3$

ii) $\forall \overline{x}, \overline{y} \in \mathbb{R}^3$, $\lambda\overline{x} + \overline{y} \in \mathbb{R}^3$ ya que $(\underset{\in \mathbb{R}}{\lambda x_1 + y_1}, \underset{\in \mathbb{R}}{\lambda x_2 + y_2}, \underset{\in \mathbb{R}}{\lambda x_3 + y_3}) \in \mathbb{R}^3$

k) $S_{11} = \{(x,y,z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2\}$

i) $\overline{O}_E \in S_{10}$? $\overline{O}_E = (0,0,0)$; $0^2 = 0^2 + 0^2$ ✓ $\Rightarrow$ $\overline{O}_E \in S_{11}$ ■

ii) $\overline{x} = (1,1,2)$, $\overline{y} = (0,2,4)$, $\lambda = 2 \in \mathbb{R}$, $\lambda\overline{x} + \overline{y} = (2,4,8)$ ¿verifica $z^2 = x^2 + y^2$?

$8^2 = 64 \overset{?}{=} 2^2 + 4^2 = 20$ ✗ no verifica, luego S_{11} no es subesv de $\mathbb{R}^3$

l) $S_{12} = \{(x,y,z) \in \mathbb{R}^3 \mid z^2 = x^2 - y^2\}$

i) $\overline{O}_E \overset{?}{\in} S_{12}$, $0^2 = 0^2 - 0^2$ ✓ $\Rightarrow$ $\overline{O}_E \in S_{12}$ ■

ii) Sean $\overline{x}, \overline{y} \in S_{11}$, $\overline{x} = (1,1,0)$, $\overline{y} = (3,2,5)$, $\lambda = 2 \in \mathbb{R}$, $\lambda\overline{x} + \overline{y} = (5,4,5)$

vemos que $5^2 \neq 5^2 - 4^2$, luego $\lambda\overline{x} + \overline{y} \notin S_{11}$ y S_{11} no es subesv de $\mathbb{R}^3$

3) Sea $(E = \{ f : \mathbb{R} \to \mathbb{R} \mid f$ es aplicación $\}, +, \cdot)$, demuestre que es e.v.

i) Asociativa $+$: Sean $f, g, h \in E$, nótese que $f(x), g(x), h(x) \in \mathbb{R}$, $\forall x \in \mathbb{R}$

$$f+(g+h) = (f+(g+h))(x) = f(x)+(g(x)+h(x)) \underset{\uparrow}{=} (f(x)+g(x))+h(x) = ((f+g)+h)(x) = (f+g)+h \quad \blacksquare$$

propiedad asociativa de la suma en el cuerpo $\mathbb{R}$.

ii) $\exists \bar{0}_E \in E$?, $\quad 0(x) : \mathbb{R} \to \mathbb{R}$, $x \to 0$, $\quad x, 0 \in \mathbb{R}$, Sea $\bar{0}_E = 0(x)$, nótese que:

$$f+0 = (f+0)(x) = f(x)+0(x) \underset{\uparrow}{=} f(x) = f \quad \blacksquare \Rightarrow \exists \bar{0}_E \in E, \text{ tq } \bar{0}_E = 0(x) \text{ (función cero)}$$

ya que $f(x) \in \mathbb{R}$, $0 \in \mathbb{R}$ y $\forall x \in \mathbb{R}$, $x+0 = x$, $\exists 0_E = 0 \in \mathbb{R}$

iii) $\exists (-f)_I \in E$? Sea $(-f) : \mathbb{R} \to \mathbb{R}$, $x \to -x$, nótese que:

$$f+(-f) = (f+(-f))(x) = f(x)+(-f)(x) \overset{\exists (-x) \text{ elem inverso } \in \mathbb{R}}{=} x+(-x) =$$

$= 0 = 0(x) \quad \blacksquare$ luego $\exists (-f)_I$ elemento inverso de E.

iv) Conmutativa $+$: Sean $f, g \in E$:

$$f+g = (f+g)(x) = f(x)+g(x) \overset{\text{conmutativa} \in \mathbb{R}}{=} g(x)+f(x) = (g+f)(x) = g+f \quad \blacksquare$$

v) Asociativa $\cdot$: Sean $\lambda, \mu \in \mathbb{R}$, $f \in E$, nótese que

$$\lambda \cdot (\mu \cdot f) = (\lambda \cdot (\mu \cdot f))(x) = \lambda \cdot (\mu \cdot f(x)) \underset{\uparrow}{=} (\lambda \cdot \mu) \cdot f(x) = ((\lambda \cdot \mu) \cdot f)(x) = (\lambda \cdot \mu) \cdot f \quad \blacksquare$$

asociativa $\cdot$ de los $\mathbb{R}$

vi) $\exists$ elemento neutro $\cdot$: Sea $f_1 : \mathbb{R} \to \mathbb{R}$, $x \to 1$, $x, 1 \in \mathbb{R}$, nótese que, para $f, f_1 \in E$,

$$f \cdot f_1 = (f \cdot f_1)(x) = f(x) \cdot f_1(x) \underset{\uparrow}{=} f(x) = f \quad \blacksquare \quad \text{luego } f_1 \text{ es elem. neutro de } E$$

ya que $\forall f(x) \in \mathbb{R}$, $f(x) \cdot 1 = f(x)$, $\exists 1$ elem neutro $\in \mathbb{R}$.

vii) Distrib. $(\bar{x}+\bar{y})\lambda$: sean $f, g \in E$, $\lambda \in \mathbb{R}$

$$(f+g) \cdot \lambda = ((f+g)\lambda)(x) = (f(x)+g(x)) \cdot \lambda \underset{\uparrow}{=} f(x) \cdot \lambda + g(x) \cdot \lambda = (f\lambda + g\lambda)(x) = f\lambda + g\lambda \quad \blacksquare$$

distrib $\mathbb{R}$

viii) Distrib $(\lambda+\mu) \cdot \bar{x}$: sea $f \in E$, $\lambda, \mu \in \mathbb{R}$:

$$(\lambda+\mu) \cdot f = ((\lambda+\mu) f)(x) = (\lambda+\mu) f(x) \underset{\uparrow}{=} \lambda f(x) + \mu f(x) = (\lambda f + \mu f)(x) = \lambda f + \mu f \quad \blacksquare$$

distrib $\mathbb{R}$

3b) a) $F = \{ f \in E \mid f(x^2) = (f(x))^2 \}$, nótese que $\bar{O}_E \in E$, $\bar{O}_E = \bar{O}(x) : \mathbb{R} \to \mathbb{R}$, $x \to 0$

i) $\bar{O}_E \in F$? $\quad O(x^2) = 0 \overset{?}{=} (O(x))^2 = 0^2 = 0$ ✓ ■ $\quad \bar{O}_E \in F$.

ii) $\lambda f + g \in F$, $\forall f, g \in F, \forall \lambda \in \mathbb{R}$, sean $f: \mathbb{R} \to \mathbb{R}$, $x \to x$, $g: \mathbb{R} \to \mathbb{R}$, $x \to 1$, $\lambda = 2$, $\left\{ \begin{array}{l} f, g \in F \\ \lambda \in \mathbb{R} \end{array} \right.$

$$\left.\begin{array}{l} (\lambda f + g)(x^2) = \lambda f(x^2) + g(x^2) = 2x^2 + 1 \\ ((\lambda f + g)(x))^2 = (\lambda f(x) + g(x))^2 = (2x+1)^2 = 4x^2 + 1 + 4x \end{array}\right\} \neq \Rightarrow \lambda f + g \notin F$$

. luego F no es subev. de E

b) $G = \{ f \in E \mid f(0) = f(2) \}$

i) $\bar{O}_E \overset{?}{\in} G \quad O(0) = 0 \overset{?}{=} O(2) = 0$ ✓ $\Rightarrow \bar{O}_E \in G$ ■

ii) Sea $f: \mathbb{R} \to \mathbb{R}$, $x \to 1$, $g: \mathbb{R} \to \mathbb{R}$, $x \to 2$, $\lambda = 2$, $\left\{ \begin{array}{l} f, g \in G \\ \lambda \in \mathbb{R} \end{array} \right.$

$(\lambda f + g)(0) = \lambda f(0) + g(0) = 2 \cdot 1 + 2$

$(\lambda f + g)(2) = \lambda f(2) + g(2) = 2 \cdot 1 + 2 \qquad$ Al parecer se cumple, pasemos a la demostración.

$\forall f, g \in G$, $\forall \lambda \in \mathbb{R}$, $\quad (\lambda f + g)(0) = \lambda \cdot f(0) + g(0) \underset{\uparrow}{=} \lambda f(2) + g(2) = (\lambda f + g)(2)$ ■

ya que $f, g \in G$ $\left\{ \begin{array}{l} f(0) = f(2) \\ g(0) = g(2) \end{array} \right.$

luego G es subev. de E

c) $H = \{ f \in E \mid f(2) = 3 + f(-1) \}$

i) $\bar{O}_E \overset{?}{\in} H$; $\quad O(2) = 0 \overset{?}{=} 3 + O(-1) = 3 + 0 = 3$ ✗ luego $\bar{O}_E \notin H$

$\Rightarrow$ H no es subev. de E

④ E - $\mathbb{R}$ev. , $F = \{e_1, e_2, e_3\} \subset E$, sist. linealmente independiente (LI)

$\bar{u}_1 = (1,1,-5)_F$ $\qquad \alpha\cdot(1,1,-5) + \beta(3,0,-1) + \gamma(1,-1,1) = 0 \overset{?}{\Rightarrow} \alpha,\beta,\gamma = 0$

$\bar{u}_2 = (3,0,-1)_F$

$\bar{u}_3 = (1,-1,1)_F$

$$\begin{aligned} \alpha + 3\beta + \gamma &= 0 \\ \alpha - \gamma &= 0 \\ -5\alpha - \beta + \gamma &= 0 \end{aligned} \qquad \left\{ \begin{aligned} &\alpha = \gamma & & \alpha = 0 \\ &3\beta + 2\gamma = 0 \Leftrightarrow \beta = \frac{-2}{3}\gamma & & \beta = 0 \\ &-5\gamma + \frac{2}{3}\gamma + \gamma = \frac{-10}{3}\gamma = 0 & \Leftrightarrow & \gamma = 0 \end{aligned} \right.$$

El sistema $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ es LI

⑤ $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$; def. como $\left\{ \begin{aligned} \bar{v}_1 &= (2,1,3,1) \\ \bar{v}_2 &= (1,0,1,0) \\ \bar{v}_3 &= (3,m,0,n) \end{aligned} \right.$, para q m, n es sist. es LI o LD.

$$\alpha(2,1,3,1) + \beta(1,0,1,0) + \gamma(3,m,0,n) = 0 \Rightarrow$$

$$\Rightarrow \left\{ \begin{aligned} &2\alpha + \beta + 3\gamma = 0 & & \\ &\alpha + m\gamma = 0 & \Rightarrow\ & -3n + m\gamma = 0 \Rightarrow \gamma = 3\frac{n}{m} \\ &3\alpha + \beta = 0 & \Rightarrow\ & -9n + \beta = 0 \Rightarrow \beta = 9n \\ &\alpha + 3n = 0 & \Rightarrow\ & \alpha = -3n \end{aligned} \right. \qquad \begin{aligned} &-6n + 9n + 9\frac{n}{m} = 0 \Rightarrow \\ &\Rightarrow 3n + \frac{9n}{m} = 0 \Rightarrow 3n\left(1 + \frac{3}{m}\right) = 0 \end{aligned}$$

El sist. cumplirá $\gamma = 0, \beta = 0, \gamma = 0 \Leftrightarrow n = 0 \wedge m \neq 0$, y en este caso será LI.

Será LD cuando uno de sus vectores, por ej. $\bar{v}_3$, puede ser expresado como combinación lineal de los otros dos:

$$\alpha(2,1,3,1) + \beta(1,0,1,0) = (3,m,0,n)$$

$$\begin{aligned} 2\alpha + \beta &= 3 \\ \alpha &= m \\ 3\alpha + \beta &= 0 \\ \alpha &= n \end{aligned} \qquad \left\{ \begin{aligned} &m = n = \alpha \\ &2m + \cancel{\beta} - 3 = 3m + \cancel{\beta} \\ &\qquad m = -3 = n \Rightarrow \left\{ \begin{aligned} \beta &= -3(-3) = 9 \\ \alpha &= -3 \end{aligned} \right. \end{aligned} \right.$$

$$-3(2,1,3,1) + 9(1,0,1,0) = (3,-3,0,-3) \checkmark \text{, para } m = n = -3$$

Será L.D. cuando $m = n = -3$.

6) $\bar{A} \in \mathbb{R}^2$, $\bar{A} \neq \bar{0}$, sea $H = \langle \bar{A} \rangle_{\mathbb{R}}$ y el conjunto

$$L = \{ \bar{x} \in \mathbb{R}^2 / \bar{x} \notin H \} \cup \{ \bar{0} \}$$

Obs./

$\bar{A} = (x, y)$ tq $x, y \neq 0$, $x, y \in \mathbb{R}$

$H = \{ \bar{A} \in \mathbb{R}^2 \mid \bar{A} = \alpha (x, y) = \alpha \cdot \bar{A}, \forall \alpha \in \mathbb{R}, x, y \neq 0 \}$

Nótese que $\bar{0} \in H$, ya que para $\alpha = 0$, $\alpha (x, y) = \bar{0}$, aunque $x, y \neq 0$.

$L = \{ \bar{x} \in \mathbb{R}^2 \mid \bar{x} \notin H \} \cup \{ \bar{0} \}$, es decir, L contiene todos los vectores que no contiene H, además del vector $\bar{0}$.

Supongamos que $\bar{A} = (x, y)$, tiene x, y determinados diferentes de 0, es decir, $x = \lambda$, $y = \mu$, $\lambda, \mu \in \mathbb{R}$

$\langle \bar{A} \rangle = \alpha (\lambda, \mu) = (\alpha\lambda, \alpha\mu)$, $\forall \alpha \in \mathbb{R}$, $\lambda, \mu \neq 0$ con valor real concreto.

Entonces L puede definirse como:

$$L = \{ \bar{x} \in \mathbb{R} \mid \bar{x} \neq (\alpha\lambda, \alpha\mu), \forall \alpha \in \mathbb{R}, \lambda, \mu \in \mathbb{R} \} \cup \{ \bar{0} \}$$

Vemos que L contiene $\bar{0}$, ¿cumple también la cerradura sobre las op. interna y externa?

Sean $\bar{v}, \bar{w} \in L$ $(\theta \bar{v} + \bar{w}) = (\theta v_1 + w_1, \theta v_2 + w_2)$

¿Es posible que $\begin{cases} \theta v_1 + w_1 = \alpha\lambda \\ \theta v_2 + w_2 = \alpha\mu \end{cases}$, al mismo tiempo que $\begin{cases} w_1, v_1 \neq \alpha\lambda \\ w_2, v_2 \neq \alpha\mu \end{cases}$?

Sea $\theta = 2$, $v_1 = 1$ y $w_1 = 2$: $2 \cdot 1 + 2 = 4 = \alpha\lambda$, para $\alpha = 1 \Rightarrow \lambda = 4$, $\mu = 7$

$v_2 = 3$, $w_2 = 1$: $2 \cdot 3 + 1 = 7 = \alpha\mu$

Vemos que $v_1 = 1, w_1 = 2 \neq \lambda \cdot \alpha = 1 \cdot 4 = 4$, pero $\theta v_1 + w_1 = 2 \cdot 1 + 2 = \lambda \cdot \alpha = 4$

y $v_2 = 3, w_2 = 1 \neq \lambda \cdot \beta = 1 \cdot 7 = 7$, pero $\theta v_2 + w_2 = 2 \cdot 3 + 1 = \lambda \cdot \beta = 7$

Demostramos así mediante un contraejemplo de que L no cumple la cerradura sobre la op. interna y externa.

Por tanto, L no es subesp. de $\mathbb{R}^2$

MDL

sólo serviría si se dijera $\forall \bar{A}$, pero como no se dice, hay que hacerlo en general $(\bar{A} = (a, b))$

7) a) Sea el sist. en $\mathbb{R}^3$: $\{\bar{e}_1=(3,0,-3),\ \bar{e}_2=(-1,1,2),\ \bar{e}_3=(4,2,-2),\ \bar{e}_4=(2,1,1)\}$ ¿Es L.I / L.D?

Este apartado contiene errores

$$\begin{cases} 3\alpha-\beta+4\gamma+2\theta=0 \\ \beta+2\gamma+\theta=0 \\ -3\alpha+2\beta-2\gamma+\theta=0 \end{cases}$$

$$\begin{cases} \theta=\theta \\ \beta=-2\gamma-\theta \\ 3\alpha+2\gamma+\theta+4\gamma+2\theta=3\alpha+6\gamma+3\theta=0\ ; \\ \alpha=\dfrac{-6\gamma-3\theta}{3}=-3\gamma-\theta \\ 9\gamma+3\theta-4\gamma-2\theta-2\gamma+\theta=\cancel{3\gamma}+2\theta=0 \end{cases}$$

$\Rightarrow \cancel{\begin{cases} \gamma=\frac{-2}{3}\theta \\ \theta=\theta \\ \alpha=2\theta-\theta=\theta \\ \beta=\frac{4}{3}\theta-\theta=\frac{1}{3}\theta \end{cases}}$

Vemos que $\forall\theta\in\mathbb{R},\ \theta\neq0$, podemos emplear coeficientes $\neq0$ para expresar el vector 0, luego el sist. es L.D.

e3 es combi lineal de e1 y e2, sin embargo e1 e2 y e4 son LI y pueden generar R3

b) En $\mathbb{R}^3$ es base? $\{\bar{u}_1,\bar{u}_2,\bar{u}_3\}$ $\begin{cases} \bar{u}_1=(1,0,-1) \\ \bar{u}_2=(1,2,1) \\ \bar{u}_3=(0,-3,2) \end{cases}$

i) ¿son LI.?

$$\begin{cases} \alpha+\beta=0 & ;\ \alpha=-\beta \\ 2\beta-3\gamma=0 & ;\ \gamma=\frac{2}{3}\beta \\ -\alpha+\beta+2\gamma=0 & ;\ \beta+\beta+\frac{4}{3}\beta=0 \Rightarrow \frac{10}{3}\beta=0\ ;\ \beta=0 \end{cases}$$

$\alpha=0 \Leftarrow \gamma=0 \Leftarrow \beta=0$

✓ son L.I.

ii) ¿Generan todo $\mathbb{R}^3$?

Este apartado contiene errores

$$\begin{cases} \alpha+\beta=x & ;\ \alpha=x-\beta \\ 2\beta-3\gamma=y & ;\ \gamma=\dfrac{y-2\beta}{-3}=\dfrac{-y+2\beta}{3}=\frac{-1}{3}y+\frac{2}{3}\beta \\ -\alpha+\beta+2\gamma=z & ;\ -x+\beta+\beta-\frac{2}{3}y+4\beta=z\ ;\quad z+x+\frac{2}{3}y=6\beta\ ;\quad \beta=\frac{1}{6}z+\frac{1}{6}x+\frac{1}{9}y \end{cases}$$

$\alpha=\frac{5}{6}x-\frac{1}{6}z-\frac{1}{9}y$ ✓

$\gamma=\frac{1}{9}z+\frac{1}{9}x+\frac{-7}{27}y$

luego $\forall x,y,z\in\mathbb{R}$, $\forall\bar{x}=(x,y,z)\in\mathbb{R}^3$, $\exists\ \alpha,\beta,\gamma$ coeficientes del sist. $\langle\bar{u}_1,\bar{u}_2,\bar{u}_3\rangle$ que generan, por tanto, todo $\mathbb{R}^3$. ■

Para $(1,0,0) \rightsquigarrow \alpha=5/6,\ \beta=1/6,\ \gamma=\frac{1}{9} \rightsquigarrow$ luego $(1,0,0)=(5/6,1/6,1/9)_{(u_1,u_2,u_3)}$

Para $(0,1,0) \rightsquigarrow \alpha=-1/9,\ \beta=1/9,\ \gamma=-7/27 \rightsquigarrow$ luego $(0,1,0)=(-1/9,1/9,-7/27)_{(u_1,u_2,u_3)}$

Para $(0,0,1) \rightsquigarrow \alpha=-1/6,\ \beta=1/6,\ \gamma=1/9 \rightsquigarrow$ luego $(0,0,1)=(-1/6,1/6,1/9)_{(u_1,u_2,u_3)}$

c) En $\mathbb{R}^4$, ¿son L.I. $\{\bar{u}_1=(1,1,2,4),\ \bar{u}_2=(2,-1,-5,2),\ \bar{u}_3=(1,-1,-4,0),\ \bar{u}_4=(2,1,1,6)\}$?

$$\begin{cases} \alpha+2\beta+\gamma+2\theta=0 \\ \alpha-\beta-\gamma+\theta=0 \\ 2\alpha-5\beta-4\gamma+\theta=0 \\ 4\alpha+2\beta+6\theta=0 \end{cases} \qquad \begin{cases} -\frac{4}{3}\theta-\frac{1}{3}\gamma+3\theta-\frac{8}{3}\theta-\frac{2}{3}\gamma-\gamma+\theta=0 \Leftrightarrow \gamma=0 \\ \alpha=-\theta-4\gamma-15\theta-10\alpha\ ;\quad \alpha=\frac{-16\theta-4\gamma}{12}=-\frac{4}{3}\theta-\frac{1}{3}\gamma \\ \beta=-3\theta-2\alpha \end{cases}$$

$$-\frac{4}{3}\theta-\overset{\nearrow 0}{\frac{1}{3}\gamma}-6\theta+\frac{8}{3}\theta+\overset{\nearrow 0}{\frac{2}{3}\gamma}+\overset{\nearrow 0}{\gamma}+2\theta=0 \quad \Leftrightarrow \quad \theta=0$$

$\gamma=0,\ \theta=0 \Rightarrow \alpha=0 \Rightarrow \beta=0$

luego $\{u_1,u_2,u_3,u_4\}$ es L.I.

---

d) $F=\{p(x)\in\mathbb{R}_2[x] \mid p(2)+p'(1)+2p''(0)=0\}$ ¿Es subev. de $\mathbb{R}_2[x]$?

i) $\bar{0}_E\in F$? $\quad \mathbb{P}_2 \quad O(x)=0+0\cdot x+0\cdot x^2=0$ ← para $a_0,a_1,a_2=0$

$\mathbb{P}_2 \quad O(x)=0+0x+0x^2+0x^3=0$

$O(2)+O'(1)+2\cdot O''(0)=0$ ✓ luego $\bar{0}_E\in F$. ■

ii) Para $\bar{x},\bar{y}\in F$, $\forall\lambda\in\mathbb{R}$, ¿$\lambda\bar{x}+\bar{y}\in F$?

Sean $\begin{cases}\bar{x}=p_1(x)=a_0+a_1x+a_2x^2 \in F \\ \bar{y}=p_2(x)=b_0+b_1x+b_2x^2 \in F\end{cases}$, $\lambda\bar{x}+\bar{y}=(\lambda a_0+\lambda a_1x+\lambda a_2x^2)+b_0+b_1x+b_2x^2$

Nótese que para $x=2$:

$(\lambda\bar{x}+\bar{y})(2)=(\lambda a_0+b_0)+(\lambda a_1+b_1)x+(\lambda a_2+b_2)x^2=$

$=\lambda a_0+b_0+2\lambda a_1+2b_1+4\lambda a_2+b_2 4$

$(\lambda\bar{x}+\bar{y})'(1)=\lambda a_1+b_1+(2\lambda a_2+2b_2)x=\lambda a_1+b_1+2\lambda a_2+2b_2$

$2(\lambda\bar{x}+\bar{y})''(0)=4\lambda a_2+4b_2$

$$\begin{cases} p_1(2)=a_0+a_1 2+a_2 4 \\ p_1'(1)=a_1+2a_2x \underset{x=1}{=} a_1+2a_2 \\ 2p_1''(0)=4a_2 \end{cases}$$

$p_1(2)+p_1'(1)+2p_1''(0)=0 \overset{*}{=} a_0+a_1 2+a_2 4+a_1+2a_2+4a_2=a_0+3a_1+10a_2$

$p_2(2)+p_2'(1)+2p_2''(0)=0=b_0+3b_1+10b_2$

Tenemos que $(\lambda\bar{x}+\bar{y})(2)+(\lambda\bar{x}+\bar{y})'(1)+(\lambda\bar{x}+\bar{y})''(0)\overset{?}{=}\lambda a_0+b_0+3\lambda a_1+3b_1+10\lambda a_2+10b_2=$

$\lambda\underbrace{(a_0+3a_1+10a_2)}_{*=0}+\underbrace{b_0+3b_1+10b_2}_{*=0}=0$ ✓ $\Rightarrow \lambda\bar{x}+\bar{y}\in F \Rightarrow F$ subev. de $\mathbb{R}_2[x]$

■

Recuerda: $\boxed{\text{dim subev} = \text{dim ev} - \text{implícitas}}$

Sabemos que $p(x)\in F$ ha de cumplir que $a_0+3a_1+10a_2=0$, o lo que es lo mismo $a_0=-3a_1-10a_2$

$p(x)=-3a_1-10a_2+a_1x+a_2x^2=a_1(-3+x)+a_2(-10+x^2) \Leftrightarrow$ base $\boxed{V=\{(-3+x),(-10+x^2)\},\ \dim_{\mathbb{R}}F=2}$

(9) Sean F, G sub ev. de $\mathbb{R}^3$,

$F = \langle (1,0,-1), (1,1,0), (0,1,1) \rangle_{\mathbb{R}}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \xrightarrow{+f_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Vemos que $\bar{v}_3 + \bar{v}_1 = \bar{v}_2$, luego podemos reexpresar F como:

$\Rightarrow F = \langle (1,0,-1), (0,1,1) \rangle_{\mathbb{R}}$, sist. L.I.

$G = \langle (2,1,-1), (1,2,1) \rangle_{\mathbb{R}}$, sist. L.I.

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix} \xrightarrow[+f_2]{-2f_2} \begin{pmatrix} 0 & -3 \\ 1 & 2 \\ 0 & 3 \end{pmatrix} \xrightarrow{+f_1} \begin{pmatrix} 0 & -3 \\ 1 & 2 \\ 0 & 0 \end{pmatrix}$$

Vemos que $\dim F = \dim G$, ¿generan los vectores generadores de G, los vectores generadores de F?
i.e.: $F \overset{?}{\subset} G$

$\alpha(2,1,-1) + \beta(1,2,1) = (1,0,-1) \Rightarrow \begin{cases} 2\alpha + \beta = 1 \\ \alpha + 2\beta = 0 \\ -\alpha + \beta = -1 \end{cases} \begin{cases} -4\beta + \beta = 1 \Rightarrow \beta = -1/3 \\ \alpha = -2\beta \Rightarrow \alpha = 2/3 \end{cases}$ ✓

$\alpha(2,1,-1) + \beta(1,2,1) = (0,1,1) \Rightarrow \begin{cases} 2\alpha + \beta = 0 \\ \alpha + 2\beta = 1 \\ -\alpha + \beta = 1 \end{cases} \begin{cases} \alpha = 1 - 2\beta \\ 2\beta + \beta = 1 \end{cases} \begin{cases} 2 - 4\beta + \beta = 0 \Rightarrow \beta = 2/3 \\ \alpha = -1/3 \end{cases}$ ✓

luego $F \subset G$, $\dim F = \dim G \Leftrightarrow F = G$

¿Pertenece $(1,-1,0)$ a F?

$\alpha(1,0,-1) + \beta(0,1,1) = (1,-1,0) \begin{cases} \alpha = 1 \\ \beta = -1 \\ -\alpha + \beta \neq 0 \end{cases}$ contradicción, luego $(1,-1,0) \notin F$

¿Pertenece $(4,5,1)$ a F?

$\alpha(1,0,-1) + \beta(0,1,1) = (4,5,1) \begin{cases} \alpha = 4 \\ \beta = 5 \\ -\alpha + \beta = 1 = -4+5 \end{cases}$ ✓ $(4,5,1) \in F$

Ec. implícita de F: $\alpha(1,0,-1) + \beta(0,1,1) = (x,y,z) \begin{cases} \alpha = x \\ \beta = y \\ -\alpha + \beta = z \end{cases}$ $\boxed{-x + y = z}$

$$\begin{pmatrix} 4 & 5 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{-5f_2} \begin{pmatrix} 4 & 0 & -4 \\ 0 & 1 & 1 \end{pmatrix}$$

vemos que $\begin{cases} (4,5,1) \in F \\ (0,1,1) \in F \end{cases}$ son L.I. luego dado que F tiene dim 2,

$\langle (4,5,1), (0,1,1) \rangle_{\mathbb{R}} = F$, $\{(4,5,1), (0,1,1)\}$ son base de F.

(10) a) Sea $\mathbb{R}^3$, $L = \langle (0,1,1), (4,1,-1), (2,1,0) \rangle_{\mathbb{R}}$, buscar base de L.

$$\begin{pmatrix} 0 & 1 & 1 \\ 4 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix} \xrightarrow{-2f_3} \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 2 & 1 & 0 \end{pmatrix} \xrightarrow{+f_1} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

Vemos que

$(4,1,-1) = 2(2,1,0) - (0,1,1)$, luego una base de L sería:

$L = \langle (0,1,1), (2,1,0) \rangle$, $\{(0,1,1), (2,1,0)\}$ es base de L

b) $\alpha(0,1,1) + \beta(2,1,0) = (x,y,z) \rightsquigarrow \begin{cases} \cancel{\alpha}^{0} \; 2\beta = x \\ \alpha + \beta = y \\ \alpha = z \end{cases} \Rightarrow z + \beta = y \Rightarrow \beta = y - z$

$2y - 2z = x \Rightarrow \boxed{x - 2y + 2z = 0}$

$(x,y,z) \in \mathbb{R}^3 \Leftrightarrow x - 2y + 2z = 0$

c) Sea $F = \{(a+3b-c,\ b-c,\ a+2b),\ \forall a,b,c \in \mathbb{R}\}$ Encuentra una base

$$\begin{cases} a + 3b - c = x \\ b - c = y \\ a + 2b = z \end{cases} \qquad \begin{cases} b = y + c \\ a = z - 2y - 2c \\ z - 2y - \cancel{2c} + 3y + \cancel{3c} - \cancel{c} = x \end{cases}$$

$\boxed{x + y - z = 0}$

$\{(2,2,0), (0,2,-2)\}$ sería base de F, son L.I. y son dos ($\dim_{\mathbb{R}} F = 2$), ya que $F \subset \mathbb{R}^3$ y tiene una implícita.

Comprobación

¿genera $\forall \bar{x} \in F$?

$\alpha(2,2,0) + \beta(0,2,-2) = (x,y,z)$

$$\begin{cases} 2\alpha = x \\ 2\alpha + 2\beta = y \\ -2\beta = z \end{cases}$$

$\rightarrow \alpha = \frac{x}{2}$

$x + 2\beta = y \rightsquigarrow \beta = \frac{y-x}{2}$

$-2\left(\frac{y-x}{2}\right) = z \Rightarrow -y + x = z \Rightarrow \boxed{x - y - z = 0}$ ✓

d) $G = L \cap F$ tq $\bar{x} \in G$ cumple $\begin{cases} x - 2y - 2z = 0 \\ x - y - z = 0 \end{cases}$ $\begin{cases} x = x \\ -y = z - x \Rightarrow y = x - z \end{cases}$

$\hookrightarrow x - 2x \cancel{+2z} - 2z = 0 \ ; \ -x - 4z = 0$

Obs / 2 ec., 3 incog. $\Rightarrow$ sólo 1 param independiente

$$\hookrightarrow \begin{cases} z = \frac{-1}{4}x \\ y = \frac{5}{4}x \\ x = x \end{cases}$$

$\forall \bar{x} \in G$, $\bar{x} = (x, \frac{5}{4}x, \frac{-1}{4}x) \Rightarrow x(1, \frac{5}{4}, \frac{-1}{4}) \Rightarrow \{(1, \frac{5}{4}, \frac{-1}{4})\}$ es base de G

10b e) $H = L + F \Rightarrow H = \langle (0,1,1), (2,1,0), (2,2,0), (0,2,-2) \rangle$

$$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -2 \end{pmatrix} \begin{matrix} \\ -f_1 \\ \\ \\ \end{matrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -2 \end{pmatrix} \begin{matrix} \\ \\ -f_2 \\ -2f_2 \end{matrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix} \begin{matrix} \\ \\ \\ +2f_3 \end{matrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \left.\begin{matrix} - \\ - \\ - \end{matrix}\right\} \text{L.I.} \\ \end{matrix}$$

$\rightarrow (0,2,-2)$ es comb. lineal de los otros tres.

$H = \langle (0,1,1), (2,1,0), (2,2,0) \rangle$, $\{(0,1,1), (2,1,0), (2,2,0)\}$ es base de H

(11) ▶ En $\mathbb{R}^4$ Este apartado contiene errores

$F_1 = \langle (1,3,-2,4), (2,5,-3,6) \rangle \Rightarrow \{(1,3,-2,4),(2,5,-3,6)\}$ base de F_1, $\dim_{\mathbb{R}} F_1 = 2$

$$\left(\begin{array}{cc|c} 1 & 2 & x \\ 3 & 5 & y \\ -2 & -3 & r \\ 4 & 6 & t \end{array}\right) \begin{array}{c} \\ -3f_1 \\ +2f_1 \\ -4f_1 \end{array} \left(\begin{array}{cc|c} 1 & 2 & x \\ 0 & -1 & y-3x \\ 0 & 1 & r+2x \\ 0 & -2 & t-4x \end{array}\right) \begin{array}{c} \\ \\ +f_2 \\ -2f_2 \end{array} \left(\begin{array}{cc|c} 1 & 2 & x \\ 0 & -1 & y-3x \\ 0 & 0 & -x+y+r \\ 0 & 0 & t-y+2x \end{array}\right) \Rightarrow \begin{cases} 2x-y+t=0 \\ -x+y+r=0 \end{cases}, \forall x,y,r,t \in \mathbb{R}$$

Obs/ F_1 tiene dimensión 2.
(en $\mathbb{R}^4$, con 2 implícitas)

$F_2 = \{(x,y,r,t) \in \mathbb{R}^4 \mid 2x-y+4r+2t=0,\ 3x+y+5r=0\} \Rightarrow \begin{cases} 2x-y+4r+2t=0 \\ 3x+y+5r=0 \end{cases}$ Obs/ F_2 tiene dim 2. (análogamente)

$$\begin{cases} 2x-y+4r+2t=0 \\ 3x+y+5r=0 \end{cases} \quad \begin{cases} t=t \\ r=r \\ x=-t-2r+\frac{1}{2}y \\ -3t-6r+\frac{3}{2}y+y+5r=0 \end{cases}$$

$x=-t-2r+\frac{3}{5}t+\frac{1}{5}r = -\frac{2}{5}t-\frac{9}{5}r$

$-3t-r+\frac{5}{2}y=0$; $y=\frac{2}{5}3t+\frac{2}{5}r = \frac{6}{5}t+\frac{2}{5}r$

$\rightsquigarrow \left(-\frac{2}{5}t-\frac{9}{5}r, \frac{6}{5}t+\frac{2}{5}r, r, t\right) \rightsquigarrow t\left(-\frac{2}{5}, \frac{6}{5}, 0, 1\right) + r\left(-\frac{9}{5}, \frac{2}{5}, 1, 0\right) \rightsquigarrow$

$F_2 = \langle (-2,6,0,5), (-9,2,5,0) \rangle$, $\{(-2,6,0,5),(-9,2,5,0)\}$ base de F_2, $\dim_{\mathbb{R}} F_2 = 2$

$F_1 \cap F_2 \Rightarrow \forall \bar{x} \in F_1 \cap F_2$, $\bar{x} = (x,y,r,t)$ $\quad \bar{x} = \alpha(1,3,-2,4) + \beta(2,5,-3,6) = \gamma(-2,6,0,5) + \mu(-9,2,5,0)$

$$\begin{cases} \alpha+2\beta+2\gamma+9\mu=0 \\ 3\alpha+5\beta-6\gamma-2\mu=0 \\ -2\alpha-3\beta-5\mu=0 \\ 4\alpha+6\beta-5\gamma=0 \end{cases} \quad \begin{cases} \alpha=-2\beta-2\gamma-9\mu \\ -6\beta-6\gamma-27\mu+5\beta-6\gamma-2\mu=0; \ -12\gamma-\beta-29\mu=0; \ \beta=-29\mu-12\gamma \\ 4\beta+4\gamma+18\mu+87\mu+36\gamma-5\mu=0; \ -116\mu-48\gamma+4\gamma+18\mu+87\mu+36\gamma-5\mu=0; \\ -16\mu-8\gamma=0; \quad \gamma=-2\mu \end{cases}$$

Sea $\mu=\mu$; $\gamma=-2\mu$
$\beta=-29\mu+24\mu=-5\mu$
$\alpha=10\mu+4\mu-9\mu=5\mu$

$(x,y,r,t) = 5\mu(1,3,-2,4) + (-5)\mu(2,5,-3,6) = (-5,-10,5,-10)\mu = \frac{\mu}{5}(-1,-2,1,-2) \Rightarrow \{(-1,-2,1,-2)\}$ es base de $F_1 \cap F_2$

Por Grassman: $\dim(F+G) = \dim F + \dim G - \dim F \cap G = 2+2-1 = 3$

$$\begin{pmatrix} 1 & 3 & -2 & 4 \\ 2 & 5 & -3 & 6 \\ -2 & 6 & 0 & 5 \\ -9 & 2 & 5 & 0 \end{pmatrix} \begin{array}{c} \\ -2f_1 \\ +2f_1 \\ +9f_1 \end{array} \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & -1 & 1 & -2 \\ 0 & 12 & -4 & 13 \\ 0 & 29 & -13 & 36 \end{pmatrix} \begin{array}{c} \\ \\ +12f_2 \\ +29f_2 \end{array} \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 8 & -11 \\ 0 & 0 & 16 & -22 \end{pmatrix} \begin{array}{c} \\ \\ \\ -2f_3 \end{array} \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 8 & -11 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$\{(1,3,-2,4), (2,5,-3,6), (-2,6,0,5)\}$ es base de F_1+F_2

$\dim_{\mathbb{R}} F_1+F_2 = 3$

$\begin{array}{r} 29 \\ -13 \\ \hline 16 \end{array} \quad \begin{array}{r} 96 \\ -58 \\ \hline 8 \end{array} \quad \begin{array}{r} 58 \\ 36 \\ \hline 22 \end{array}$

(12) F, G, subev. de $\mathbb{R}^3$

$$F = \langle (1, \beta-2, \beta), (0,1,-1), (1, \beta, \beta-2) \rangle_{\mathbb{R}}$$
$$G = \langle (1,1,-1), (1,-1,0), (1,-3,1) \rangle_{\mathbb{R}}$$

$$\left\{ \begin{array}{l} \dim_{\mathbb{R}} F \leq 3 \\ \dim_{\mathbb{R}} G \leq 3 \end{array} \right.$$

Evaluamos LD/LI de los sist. generadores de F y G:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & -3 & 1 \end{pmatrix} \begin{matrix} \\ -f_1 \\ -f_1 \end{matrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{pmatrix} \begin{matrix} \\ \\ -2f_2 \end{matrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \{(1,1,-1), (1,-1,0)\} \text{ es base de } G, \ \dim_{\mathbb{R}} G = 2$$

$$\begin{pmatrix} 1 & \beta-2 & \beta \\ 0 & 1 & -1 \\ 1 & \beta & \beta-2 \end{pmatrix} \begin{matrix} \\ \\ -f_1 \end{matrix} \begin{pmatrix} 1 & \beta-2 & \beta \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \begin{matrix} \\ \\ -2f_2 \end{matrix} \begin{pmatrix} 1 & \beta-2 & \beta \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \{(1, \beta-2, \beta), (0,1,-1)\} \text{ es base de } F, \ \dim_{\mathbb{R}} F = 2$$

Analizamos dimensión de $F+G$:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & \beta-2 & \beta \\ 0 & 1 & -1 \end{pmatrix} \begin{matrix} \\ -f_1 \\ -f_1 \\ \\ \end{matrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & \beta-3 & \beta+1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & \beta-2 & \beta+1 \\ 0 & -2 & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ +2f_2 \end{matrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & \beta-2 & \beta+1 \\ 0 & 0 & -1 \end{pmatrix} \begin{matrix} -f_1 \\ -f_1 \\ +(\beta+1)f_1 \\ \\ \end{matrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & \beta-2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{matrix} \\ \\ -(\beta-2)f_2 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \dim_{\mathbb{R}} F+G = 3 \quad \forall \beta \in \mathbb{R},$$

Grassman: $\dim(F \cap G) = \dim F + \dim G - \dim(F+G) = 2 + 2 - 3 = 1$

(13) $\mathbb{R}_n[x]$, $n \in \mathbb{N}^*$, definimos:

$$L = \{ p(x) \in \mathbb{R}_n[x] \mid p'(1) = 0 \}$$

L es subev. $\Leftrightarrow \begin{cases} i) & \bar{0}_E \in L \\ ii) & \lambda \bar{x} + \bar{y} \in L, \ \forall \bar{x}, \bar{y} \in L \end{cases}$

i) $\bar{0}_E \in \mathbb{R}_n[x]$, $\bar{0}_E = 0(x) \overset{def}{=} 0$

$0'(1) = 0 \Rightarrow \bar{0}_E \in L$ ■

ii) Sean $\bar{x} = p_1(x) = a_0 + a_1 x + \ldots + a_n x^n$, $\bar{y} = p_2(x) = b_0 + b_1 x + \ldots + b_n x^n$ tq.

$$\left\{ \begin{aligned} p_1'(1) &= a_1 + 2a_2 + 3a_3 + \ldots + n a_n = 0 \\ p_2'(1) &= b_1 + 2b_2 + 3b_3 + \ldots + n b_n = 0 \end{aligned} \right\}, \text{ ie } p_1(x), p_2(x) \in L,$$

$$\lambda \bar{x} + \bar{y} = \lambda p_1(x) + p_2(x) = \lambda (a_0 + a_1 x + \ldots + a_n x^n) + b_0 + b_1 x + \ldots + b_n x^n = \lambda a_0 + \ldots + \lambda a_n x^n + b_0 + \ldots + b_n x^n$$

$$(\lambda \bar{x} + \bar{y})'(1) = \lambda a_1 + 2\lambda a_2 + 3\lambda a_3 + \ldots + n\lambda a_n + b_1 + 2b_2 + \ldots + n b_n =$$

$$= \lambda \underbrace{(a_1 + 2a_2 + \ldots + n a_n)}_{=0} + \underbrace{b_1 + \ldots + n b_n}_{=0} = 0 \ ■$$

$p_1(x), p_2(x) \in L \leadsto$

Luego L es subev. de $\mathbb{R}_n[x]$

$p(x) \in L \Leftrightarrow a_1 + 2a_2 + \ldots + n a_n = 0 \Rightarrow 2a_2 + \ldots + n a_n = -a_1 \Rightarrow$

$$p(x) = a_0 - (2a_2 + \ldots + n a_n) x + a_2 x^2 + \ldots + a_n x^n = a_0 + a_2(-2 + x^2) + a_3(-3 + x^3) + \ldots + a_n(-n + x^n)$$

$\Rightarrow \{1, -2 + x^2, -3 + x^3, \ldots, -n + x^n\}$ es base de L, $\dim_{\mathbb{R}} L = n$

Un espacio suplementario sería $G = \langle x \rangle_{\mathbb{R}}$, base $\{x\}$. $L \oplus G = \mathbb{R}_n[x]$

$\dim_{\mathbb{R}} G = 1$

(14)

a) H, G subev. de F $\mathbb{K}$ev. | $\dim_{\mathbb{K}} F = n$.

Demuestra $\dim_{\mathbb{K}} H + \dim_{\mathbb{K}} G > n \Rightarrow \dim(G \cap H) > 0$

H, G subev $\Rightarrow \dim(H+G) \leq \dim_{\mathbb{K}} F = n$

Por Grassman: $\dim(H \cap G) = \underbrace{\dim H + \dim G}_{>n} - \underbrace{\dim(H+G)}_{\leq n} \Rightarrow \dim(H \cap G) > 0$ ■

b) $n = \dim(H \cap G) \leq \dim H \leq \dim(H+G) = n+1$

(15) $F = \langle \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 \rangle_{\mathbb{R}}$ en $\mathbb{R}^3$, donde: $\begin{cases} \bar{x}_1 = (1,1,-1) \\ \bar{x}_2 = (3,8,2) \\ \bar{x}_3 = (1,1,-1) \\ \bar{x}_4 = (-2,-2,2) \end{cases}$

a) $\begin{pmatrix} 1 & 1 & -1 \\ 3 & 8 & 2 \\ 1 & 1 & -1 \\ -2 & -2 & 2 \end{pmatrix} \begin{matrix} \\ -3f_1 \\ -f_1 \\ +2f_1 \end{matrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$ $\{(1,1,-1),(3,8,2)\}$ es base de F, $\dim_{\mathbb{R}} F = 2$

b) $(3,-17,-23) = \alpha(1,1,-1) + \beta(3,8,2)$, $\exists ?\ \alpha, \beta \in \mathbb{R}$

$\begin{cases} \alpha + 3\beta = 3 \\ \alpha + 8\beta = -17 \\ -\alpha + 2\beta = -23 \end{cases}$ $\begin{cases} \alpha = 3 - 3\beta = 3 - (3\cdot(-4)) = 15 \\ 3 - 3\beta + 8\beta = -17 \ ;\ 3 + 5\beta = -17 \ ;\ \beta = -4 \\ -15 + 2\cdot(-4) = -23 \overset{\checkmark}{=} -23 \end{cases}$ $\Rightarrow \exists\, \alpha, \beta$ tq $\begin{matrix} \alpha = 15 \\ \beta = -4 \end{matrix} \in \mathbb{R}$.

luego $(3,-17,-23) \in F$. $(3,-17,-23) = (15,-4)_{\text{base } \{(1,1,-1),(3,8,2)\}}$

c) $G = \langle (0,0,1) \rangle_{\mathbb{R}}$ es espacio suplementario de F, tq $F \oplus G = \mathbb{R}^3$, nótese que

rango $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 5 & 5 \\ 0 & 0 & 1 \end{pmatrix} = 3 \Rightarrow \dim F + G = 3 \Rightarrow \dim(F \cap G) = \dim F + \dim G - \dim F + G = 0 \Rightarrow$

$F \cap G = \{\bar{0}\}$, luego L, G suplementarios.

d) $\bar{x} = (2,17,12)$ calcular descomposición única $\bar{x} = \bar{y} + \bar{a}$, con $\bar{y} \in F$, $\bar{a} \in G$:

$\bar{x} = (2,17,12)$ $\quad \bar{y} = (y_1, y_2, y_3) = \alpha(1,1,-1) + \beta(3,8,2)$

$\bar{a} = (a_1, a_2, a_3) = \gamma(0,0,1)$

$\bar{x} = (2,17,12) = \bar{y} + \bar{a} = \alpha(1,1,-1) + \beta(3,8,2) + \gamma(0,0,1) \Rightarrow$ $\begin{cases} \alpha + 3\beta = 2 \\ \alpha + 8\beta = 17 \\ -\alpha + 2\beta + \gamma = 12 \\ \hline 7 + 6 + \gamma = 12 \end{cases} \Rightarrow \gamma = -1$ $\quad \begin{cases} \alpha = 2 - 3\beta \\ 2 - 3\beta + 8\beta = 17 \\ \beta = 3 \\ \alpha = -7 \end{cases}$

$\bar{y} = (-7)(1,1,-1) + (3)(3,8,2) = (-7,-7,7) + (9,24,6) = (2,17,13)$

$\bar{a} = (-1)(0,0,1) = (0,0,-1)$

$\bar{x} = (2,17,12) = \bar{y} + \bar{a} = \underset{\in F}{(2,17,13)} + \underset{\in G}{(0,0,-1)}$

(16) F, G, H subesp. de E-es. sobre $\mathbb{K}$

$$\dim_{\mathbb{K}}(F\cap(G+H)) = \dim_{\mathbb{K}}(F\cap G) + \dim_{\mathbb{K}}(F\cap H) + \dim_{\mathbb{K}}(F\cap G\cap H)$$

Sea $G = H$ y $F\cap G \neq \{0\} \Rightarrow$

$m = \dim(F\cap(G+H)) = \dim(F\cap G) = \dim(F\cap H) = \dim(F\cap G\cap H)$, $0 < m$, $m\in\mathbb{N}^*$

$\Rightarrow$ $m \stackrel{?}{=} m+m+m$ vemos que no es posible $\Rightarrow$ contraejemplo de que no se cumple.
ya que $m \neq 0$

17) Sea $E_3 \subset \mathbb{R}^3$, E_3 e.v., $\dim_{\mathbb{R}} E_3 = 3$

$V \subset E_3$ base de E_3

Subev. de E_3: $\begin{cases} H = \{(x,y,z)_V \in E_3 \mid x+y+z=0\} \\ L = \langle (1,0,1)_V \rangle_{\mathbb{R}} \end{cases}$

a) H es subev.? i) $\bar{0}_E \in H$? $\bar{0}_E = (0,0,0) = \bar{0}$, vemos q $0+0+0=0$, luego $\bar{0}_E \in H$

ii) $\lambda\bar{x}+\bar{y} \in H$? $\forall \bar{x},\bar{y} \in H$

$\lambda\bar{x}+\bar{y} = (\lambda x_1+y_1, \lambda x_2+y_2, \lambda x_3+y_3)$

$\lambda x_1+y_1+\lambda x_2+y_2+\lambda x_3+y_3 \overset{?}{=} 0$

$\lambda \underbrace{(x_1+x_2+x_3)}_{=0} + \underbrace{(y_1+y_2+y_3)}_{=0} = 0$, luego $\lambda\bar{x}+\bar{y} \in H$

ya que $\bar{x},\bar{y} \in H$

H es subev. de E_3

b) $\bar{x} \in H$; $\bar{x} = (x,y,z)_V$ tq $x+y+z=0 \Rightarrow x = -y-z \Rightarrow \bar{x} = (-y-z, y, z)_V = y(-1,1,0) + z(-1,0,1)$

$\{(-1,1,0)_V, (-1,0,1)_V\}$ es base de H, $\dim_{\mathbb{R}} H = 2$

c) $H+L = \langle (-1,1,0)_V, (-1,0,1)_V, (1,0,1)_V \rangle_{\mathbb{R}}$

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{matrix} \\ -f_1 \\ +f_1 \end{matrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} \\ \\ +f_2 \end{matrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$\Rightarrow \{(-1,1,0), (-1,0,1), (1,0,1)\}$ es base de H

$\Rightarrow \dim_{\mathbb{R}} H = 3$

$\dim(H\cap L) = \dim H + \dim L - \dim(H+L) = 2+1-3 = 0$

$\Rightarrow H \cap L = \{\bar{0}\}$, $\{\bar{0}\}$ es base de $H\cap L$, $\dim(H\cap L) = 0$.

d) sí, ya que $H\cap L = \{\bar{0}\}$ y $H \oplus L = E_3$

$\bar{w} = (1,-1,1)_V = \underset{\in H}{\bar{w}_1} + \underset{\in L}{\bar{w}_2} = \alpha(-1,1,0) + \beta(-1,0,1) + \gamma(1,0,1) \Rightarrow \begin{cases} -\alpha-\beta+\gamma = 1 \\ \alpha = -1 \\ \beta+\gamma = 1 \end{cases}$; $-\beta+\gamma = 0$; $\gamma = \beta$

; $\beta = \gamma = 1/2$

solución única.

$\Rightarrow \bar{w}_1 = -1(-1,1,0) + \frac{1}{2}(-1,0,1) = (1,-1,0) + (-\frac{1}{2}, 0, \frac{1}{2}) = (\frac{1}{2}, -1, \frac{1}{2})$

$\bar{w}_2 = \frac{1}{2}(1,0,1) = (\frac{1}{2}, 0, \frac{1}{2})$

$\bar{w}_1+\bar{w}_2 = (\frac{1}{2},-1,\frac{1}{2}) + (\frac{1}{2},0,\frac{1}{2}) = (1,-1,1) = \bar{w}$ ✓

Solución $\bar{w}_1, \bar{w}_2$ única.

(18) $\bar{x}, \bar{y}, \bar{a} \in \mathbb{R}^4$

$\bar{x} = (x_1, x_2, x_3, x_4)$

$\bar{y} = (y_1, y_2, y_3, y_4)$

$\bar{a} = (a_1, a_2, a_3, a_4)$

$\{\bar{x}, \bar{y}, \bar{a}, \bar{t}\}$ es base de $\mathbb{R}^4 \Leftrightarrow \begin{cases} \bar{x}, \bar{y}, \bar{a}, \bar{t} \text{ son L.I.} \\ \text{generan } \mathbb{R}^4 \end{cases}$

Sabemos que: $\begin{pmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ a_1 & a_3 & a_4 \end{pmatrix}$ son L.I. , y $\overset{*}{\begin{cases} t_1 = x_1 + y_1 + a_1 \\ t_2 \neq x_2 + y_2 + a_2 \\ t_3 = x_3 + y_3 + a_3 \\ t_4 = x_4 + y_4 + a_4 \end{cases}}$

Supongamos que $\{\bar{x}, \bar{y}, \bar{a}, \bar{t}\}$ no es base de $\mathbb{R}^4 \Rightarrow \dim_{\mathbb{R}} \langle (\bar{x}, \bar{y}, \bar{a}, \bar{t}) \rangle < 4 \Rightarrow r(A) < 4$, para A:

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ a_1 & a_2 & a_3 & a_4 \\ t_1 & t_2 & t_3 & t_4 \end{pmatrix} \underset{*}{=} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ a_1 & a_2 & a_3 & a_4 \\ x_1+y_1+a_1 & t_2 & x_3+y_3+a_3 & x_4+y_4+a_4 \end{pmatrix} \xrightarrow{-f_1-f_2-f_3} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & t_2-x_2-y_2-a_2 & 0 & 0 \end{pmatrix}$$

Pero (x_1, x_3, x_4), (y_1, y_3, y_4), (a_1, a_3, a_4) son L.I , y $t_2 \neq x_2 + y_2 + a_2 \Rightarrow t_2 - x_2 - y_2 - a_2 \neq 0$, luego $r(A) = 4 \Rightarrow \dim_{\mathbb{R}} \langle (\overline{x, y, a, t}) \rangle = 4$.

Así pues, $\{\bar{x}, \bar{y}, \bar{a}, \bar{t}\}$ es base de $\mathbb{R}^4$ $\forall \bar{x}, \bar{y}, \bar{a}, \bar{t} \in \mathbb{R}^4$ que cumplan estas condiciones.

(1a) Sea B, F subesp. de $\mathbb{R}_3[x]$:

$$B = \{ 1+x^2+x^3,\ 2+x-x^2,\ 1+x-2x^2+x^3,\ 3+2x+x^2+x^3 \}$$

obs

$$F = \{ p(x) \in \mathbb{R}_3[x] \mid p'(1) = p''(1) \ \wedge \ p(-1) = 0 \}$$

Sea $V = \{1, x, x^2, x^3\}$ base canónica de $\mathbb{R}_3[x]$:

$$B = \{ (1,0,1,1)_V,\ (2,1,-1,0)_V,\ (1,1,-2,1)_V,\ (3,2,1,1)_V \}$$

Sea $\bar{x} \in F$, $\bar{x} = p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ t.q. $\begin{cases} p'(1) = p''(1), \text{ i.e., } a_1 + \cancel{2a_2} + 3a_3 = \cancel{2a_2} + 6a_3 \\ p(-1) = 0, \text{ i.e., } a_0 - a_1 + a_2 - a_3 = 0 \end{cases}$

$p'(x) = a_1 + 2a_2 x + 3a_3 x^2$

$p''(x) = 2a_2 + 6a_3 x$

$$\Rightarrow \begin{cases} a_1 - 3a_3 = 0 \\ a_0 - a_1 + a_2 - a_3 = 0 \end{cases} \quad \begin{cases} a_1 = 3a_3 ; \\ a_0 - 3a_3 + a_2 - a_3 = a_0 - 4a_3 + a_2 = 0 ; \ a_0 = -a_2 + 4a_3 ; \quad a_2 = a_2,\ a_3 = a_3 \end{cases}$$

$$\Rightarrow p(x) = (-a_2 + 4a_3) + 3a_3 \cdot x + a_2 x^2 + a_3 x^3 = a_2(-1+x^2) + a_3(4+3x+x^3)$$

$\Rightarrow F = \langle -1+x^2, 4+3x+x^3 \rangle$, $\dim_{\mathbb{R}} F = 2$, $\{-1+x^2, 4+3x+x^3\}$ es base de F.

$\{(-1,0,1,0)_V, (4,3,0,1)_V\}$ es base de F, expresada en base canónica de $\mathbb{R}_3[x]$.

---

a) B base de $\mathbb{R}_3[x] \Leftrightarrow$ i) $\dim_{\mathbb{R}} B = 4$
ii) B genera $\mathbb{R}_3[x]$.

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & -1 & 0 \\ 1 & 1 & -2 & 1 \\ 3 & 2 & 1 & 1 \end{pmatrix} \begin{matrix} \\ -2f_1 \\ -f_1 \\ -3f_1 \end{matrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -2 & -2 \end{pmatrix} \begin{matrix} \\ \\ -f_2 \\ -2f_2 \end{matrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 4 & 2 \end{pmatrix} \ f_2 \leftrightarrow f_3 \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$r(A) = 4 \Rightarrow \dim_{\mathbb{R}} B = 4$

Sea $v(x) \in \mathbb{R}_3[x]$; $v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = (a_0, a_1, a_2, a_3)_V$, veamos si B genera todo $v(x) \in \mathbb{R}_3[x]$

$$(a_0, a_1, a_2, a_3) = \alpha(1,0,1,1) + \beta(2,1,-1,0) + \mu(1,1,-2,1) + \gamma(3,2,1,1) \Rightarrow \begin{cases} \alpha + 2\beta + \mu + 3\gamma = a_0 & \text{(i)} \\ \beta + \mu + 2\gamma = a_1 & \text{(ii)} \\ \alpha - \beta - 2\mu + \gamma = a_2 & \text{(iii)} \\ \alpha + \mu + \gamma = a_3 & \text{(iv)} \end{cases}$$

(19b)

(i) $\alpha = a_0 - 2\beta - \mu - 3\gamma$

(iii) $a_0 - 2\beta - \mu - 3\gamma - \beta - 2\mu + \gamma = a_2$; $-3\beta - 2\gamma - 3\mu = a_2 - a_0$; $\beta = \dfrac{a_2 - a_0 + 3\mu + 2\gamma}{-3} = \dfrac{-1}{3}a_2 + \dfrac{1}{3}a_0 - \mu - \dfrac{2}{3}\gamma$

(ii) $\dfrac{-1}{3}a_2 + \dfrac{1}{3}a_0 - \cancel{\mu} - \dfrac{2}{3}\gamma + \cancel{\mu} + 2\gamma = a_1$; $\dfrac{-1}{3}a_2 + \dfrac{1}{3}a_0 + \dfrac{4}{3}\gamma = a_1$; $\gamma = \dfrac{3}{4}a_1 + \dfrac{3}{4}\dfrac{1}{3}a_2 - \dfrac{3}{4}\dfrac{1}{3}a_0 =$

$= \gamma = \dfrac{3}{4}a_1 + \dfrac{1}{4}a_2 - \dfrac{1}{4}a_0$

(iv) $a_0 - 2\beta - \cancel{\mu} - 3\gamma + \cancel{\mu} + \dfrac{3}{4}a_1 + \dfrac{1}{4}a_2 - \dfrac{1}{4}a_0 = \dfrac{3}{4}a_0 + \dfrac{3}{4}a_1 + \dfrac{1}{4}a_2 - 2\beta - \dfrac{9}{4}a_1 - \dfrac{3}{4}a_2 + \dfrac{3}{4}a_0 = a_3$;

$-2\beta = a_3 - \dfrac{6}{4}a_0 + \dfrac{6}{4}a_1 + \dfrac{1}{2}a_2$; $\beta = \dfrac{-1}{2}a_3 + \dfrac{3}{4}a_0 - \dfrac{3}{4}a_1 - \dfrac{1}{4}a_2$

(ii) $\dfrac{-1}{3}a_3 + \dfrac{3}{4}a_0 - \dfrac{3}{4}a_1 - \dfrac{1}{4}a_2 + \mu + \dfrac{3}{2}a_1 + \dfrac{1}{2}a_2 - \dfrac{1}{2}a_0 = a_1$;

$\mu = a_1 - \dfrac{3}{2}a_1 + \dfrac{3}{4}a_1 + \dfrac{1}{2}a_0 - \dfrac{3}{4}a_0 - \dfrac{1}{2}a_2 + \dfrac{1}{4}a_2 + \dfrac{1}{3}a_3 = \dfrac{-1}{4}a_0 - \dfrac{1}{4}a_2 + \dfrac{1}{3}a_3 + \dfrac{1}{4}a_1 = \mu$

$\alpha = a_0 + a_3 - \dfrac{6}{4}a_0 + \dfrac{6}{4}a_1 + \dfrac{2}{4}a_2 + \dfrac{1}{4}a_0 + \dfrac{1}{4}a_2 - \dfrac{1}{3}a_3 - \dfrac{1}{4}a_1 - \dfrac{9}{4}a_1 - \dfrac{3}{4}a_2 + \dfrac{3}{4}a_0 =$

$= \dfrac{2}{4}a_0 - \dfrac{4}{4}a_1 + \dfrac{2}{3}a_3 = \dfrac{1}{2}a_0 - a_1 + \dfrac{2}{3}a_3 = \alpha$

$$\begin{cases} \alpha = \frac{1}{2}a_0 - a_1 + \frac{2}{3}a_3 \\ \beta = \frac{3}{4}a_0 - \frac{3}{4}a_1 - \frac{1}{4}a_2 - \frac{1}{2}a_3 \\ \gamma = -\frac{1}{4}a_0 + \frac{1}{4}a_2 + \frac{3}{4}a_1 \\ \mu = -\frac{1}{4}a_0 + \frac{1}{4}a_1 - \frac{1}{4}a_2 + \frac{1}{3}a_3 \end{cases}$$

Luego $\forall \bar{v}(x) \in \mathbb{R}_3[x]$, $\exists \alpha, \beta, \gamma, \mu$ tq B genera $\bar{v}(x)$. ■

b) F subesp. de $\mathbb{R}_3[x] \Leftrightarrow$ i) $\bar{0}_E \in \mathbb{R}_3[x]$, $\bar{0}_E = 0(x) \in F$

ii) $\lambda \bar{x} + \bar{y} \in F$, $\forall \bar{x}, \bar{y} \in F$.

Nótese que $F = \{ p(x) \in \mathbb{R}_3[x] \mid$ si $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, $a_1 - 3a_3 = 0 \wedge a_0 - a_1 + a_2 - a_3 = 0 \}$

Vemos que $0(x)$ cumple con las dos ecuaciones implícitas de F, luego $\bar{0}_E \in F$. ■ (i)

Sean $p_1(x), p_2(x) \in F$ tales que sus coeficientes reales cumplen, respectivamente $\begin{cases} a_1 - 3a_3 = 0 \wedge a_0 - a_1 + a_2 - a_3 = 0 \\ b_1 - 3b_3 = 0 \wedge b_0 - b_1 + b_2 - b_3 = 0 \end{cases}$

$$\begin{cases} \lambda p_1(x) + p_2(x) = \lambda(a_0 + a_1x + a_2x^2 + a_3x^3) + b_0 + b_1x + b_2x^2 + b_3x^3 \\ (\lambda p_1(x) + p_2(x))' = \lambda p_1'(x) + p_2'(x) = \lambda(a_1 + 2a_2x + 3a_3x^2) + b_1 + 2b_2x + 3b_3x^2 \\ (\lambda p_1(x) + p_2(x))'' = \lambda p_1''(x) + p_2''(x) = \lambda(2a_2 + 6a_3x) + 2b_2 + 6b_3x \end{cases}$$

comprobamos que $\begin{cases} p'(1) = p''(1) & \text{(i)b} \\ p(-1) = 0 & \text{(ii)b} \end{cases}$

./..

19c

$$\lambda(a_1 + 2a_2 + 3a_3) + b_1 + 2b_2 + 3b_3 \overset{?}{=} \lambda(2a_2 + 6a_3) + 2b_2 + 6b_3 \Rightarrow$$

$$\Rightarrow \lambda(a_1 + \cancel{2a_2} + 3a_3 - \cancel{2a_2} - 6a_3) + b_1 + \cancel{2b_2} + 3b_3 - \cancel{2b_2} - 6b_3 \overset{?}{=} 0 \Rightarrow$$

$$\underbrace{\lambda(a_1 - 3a_3)}_{=0} + \underbrace{b_1 - 3b_3}_{=0} = 0 \quad \checkmark \quad \blacksquare$$

, ya que $p_1(x), p_2(x) \in F$ — luego F es subesp. de $\mathbb{R}_3[x]$

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c) $\{-1 + x^2,\ 4 + 3x + x^3\}$ es base de F, $\dim_{\mathbb{R}} F = 2$ (ver observaciones al inicio)

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d) Basta encontrar dos vectores generadores que sean L.I con respecto a los dos vectores generadores de F.

$$F = \langle (-1,0,1,0)_V,\ (4,3,0,1)_V \rangle$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ - & & & \\ - & & & \end{pmatrix} \xrightarrow{+4f_1} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \\ \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{vectores suplementarios} \end{matrix}$$

luego $G = \langle (0,0,1,0)_V,\ (0,0,0,1)_V \rangle = \langle x^2, x^3 \rangle$ es suplementario de F tal que:

$$F \oplus G = \mathbb{R}_3[x]$$

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e) Encuentre $\dim G+H$ y $G \cap H$, donde $H = \langle (4,1,-7,3)_B,\ (-4,-5,3,5)_B,\ (-12,-9,-1,13)_B \rangle$

$$\{(-4,1,-7,3)_B,\ (-4,5,3,5)_B,\ (-12,-9,-1,13)_B\}$$

$$\begin{pmatrix} -4 & 1 & -7 & 3 \\ -4 & -5 & 3 & 5 \\ -12 & -9 & -1 & 13 \end{pmatrix} \begin{matrix} \\ -f_1 \\ -3f_1 \end{matrix} \begin{pmatrix} -4 & 1 & -7 & 3 \\ 0 & -6 & 10 & 2 \\ 0 & -12 & 20 & 4 \end{pmatrix} \begin{matrix} \\ \\ -2f_2 \end{matrix} \begin{pmatrix} -4 & 1 & -7 & 3 \\ 0 & -6 & 10 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\dim_{\mathbb{R}} H = 2$

$\{(-4,1,-7,3)_B,\ (-4,-5,3,5)_B\}$ es base de H

$G = \langle (0,0,1,0)_V,\ (0,0,0,1)_V \rangle$, $\dim_{\mathbb{R}} G = 2$, $\{(0,0,1,0)_V,\ (0,0,0,1)_V\}$ es base de G

$$[Id]_{BV} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & -1 & -2 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad [x]_B = \begin{bmatrix} -4 \\ 1 \\ -7 \\ 3 \end{bmatrix}, \quad [y]_B = \begin{bmatrix} -4 \\ -5 \\ 3 \\ 5 \end{bmatrix}$$

matriz cambio de base

$$\Rightarrow [x]_V = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & -1 & -2 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 1 \\ -7 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \\ -8 \end{bmatrix}$$

$$[y]_V = \begin{bmatrix} '' & '' & '' \end{bmatrix} \begin{bmatrix} -4 \\ -5 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 0 \\ 4 \end{bmatrix}$$

$$\Rightarrow H = \langle (0,0,12,-8)_V,\ (4,8,0,4)_V \rangle$$

$$F+G = \langle (0,0,1,0)_V,\ (0,0,0,1)_V,\ (0,0,12,-8)_V,\ (4,8,0,4)_V \rangle$$

$$R \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 12 & -8 \\ 4 & 8 & 0 & 4 \end{pmatrix} = 3 \Rightarrow \dim_{\mathbb{R}}(F+G) = 3$$

$$\dim(F \cap G) = \dim F + \dim G - \dim(F+G) = 2 + 2 - 3 = 1$$

19d) Suplementario $F \cap G$, $\dim_{\mathbb{R}} F \cap G = 1$

Comprobación

$F = \{ p(x) \in \mathbb{R}_3[x] \mid p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ cumple: $a_1 - 3a_3 = 0$; $a_0 - a_1 + a_2 - a_3 = 0 \}$

$G = \{ j(x) \in \mathbb{R}_3[x] \mid j(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$ cumple: $a_3 = 0$, $a_1 = 0 \}$

$\bar{x} \in F \cap G$ cumplirá
$$\begin{cases} \cancel{a_1 - 3a_3 = 0} \\ a_0 - a_1 + a_2 - a_3 = 0 \\ a_1 = 0 \\ a_3 = 0 \end{cases} \qquad \begin{cases} a_0 - a_1 + a_2 - a_3 = 0 \Rightarrow a_0 = \cancel{a_1} - a_2 + \cancel{a_3} \\ a_1 = 0 \\ a_3 = 0 \end{cases}$$

$$\bar{x} = -a_2 + 2a_2 x^2 = a_2(-1 + 2x^2)$$

$\{-1 + 2x^2\}$ es base de $F \cap G$, $\boxed{\dim_{\mathbb{R}} F \cap G = 1}$

(20) Sea $\vec{u} \in \mathbb{R}^4$, $\vec{u}=(a,b,c,d)$

$$F_{\vec{u}} = \{(x,y,z,t) \in \mathbb{R}^4 \mid ax+by+cz+dt=0\} \subset \mathbb{R}^4$$

a) $F_{\vec{u}}$ es subev. ?

i) $\bar{0}_E \overset{?}{\in} F_{\vec{u}}$ $\quad a\cdot 0 + b\cdot 0 + c\cdot 0 + d\cdot 0 = 0$ ✓, luego $\bar{0}_E \in F_{\vec{u}}$ ■

ii) $\lambda\bar{x}+\bar{y} \overset{?}{\in} F_{\vec{u}}$, $\forall \bar{x},\bar{y} \in F_{\vec{u}}$, sean $\bar{x}=(x_1,x_2,x_3,x_4)$, $\bar{y}=(y_1,y_2,y_3,y_4)$

$$\lambda\bar{x}+\bar{y} = (\lambda x_1+y_1, \lambda x_2+y_2, \lambda x_3+y_3, \lambda x_4+y_4)$$

$$a(\lambda x_1+y_1)+b(\lambda x_2+y_2)+c(\lambda x_3+y_3)+d(\lambda x_4+y_4) \overset{?}{=} 0 \ ;$$

$$\underbrace{\lambda(ax_1+bx_2+cx_3+dx_4)}_{=0} + \underbrace{ay_1+by_2+cy_3+dy_4}_{=0} = 0$$ ✓ , luego $\lambda\bar{x}+\bar{y} \in F_{\vec{u}}$ ■

, ya que $\bar{x},\bar{y} \in F_{\vec{u}}$

$F_{\vec{u}}$ es subev. de $\mathbb{R}^4$, $\forall \vec{u} \in \mathbb{R}^4$

b) $F_{\vec{u}} = \{(x,y,z,t) \in \mathbb{R}^4 \mid x+by+cz+dt=0\} \subset \mathbb{R}^4$

$x = -by-cz-dt \Rightarrow \bar{x} \in F_{\vec{u}} = (-by-cz-dt, y, z, t) =$

$= y(-b,1,0,0) + z(-c,0,1,0) + t(-d,0,0,1)$

$\{(-b,1,0,0),(-c,0,1,0),(-d,0,0,1)\}$ es base de $F_{\vec{u}} \Rightarrow \dim_{\mathbb{R}} F_{\vec{u}} = 3$

c) $N = \{(1,0,1,0),(1,1,0,0),(1,0,0,1),(1,1,1,1)\} \subset \mathbb{R}^4$, ¿es base de $\mathbb{R}^4$?

i) L.I.:
$$\begin{pmatrix} 1&0&1&0\\1&1&0&0\\1&0&0&1\\1&1&1&1 \end{pmatrix} \begin{matrix} \\ -f_1 \\ -f_1 \\ -f_1 \end{matrix} \begin{pmatrix} 1&0&1&0\\0&1&-1&0\\0&0&-1&1\\0&1&0&1 \end{pmatrix} \begin{matrix} \\ \\ \\ -f_2 \end{matrix} \begin{pmatrix} 1&0&1&0\\0&1&-1&0\\0&0&-1&1\\0&0&1&1 \end{pmatrix} \begin{matrix} \\ \\ \\ +f_3 \end{matrix} \begin{pmatrix} 1&0&1&0\\0&1&-1&0\\0&0&-1&1\\0&0&0&2 \end{pmatrix}$$ ✓ $\{u_1,u_2,u_3,u_4\}$ son L.I.

ii) ¿Generan todo $\mathbb{R}^4$? $\quad \forall \bar{x}=(x,y,z,t) \in \mathbb{R}^4$:

$$(x,y,z,t) = \alpha(1,0,1,0)+\beta(1,1,0,0)+\gamma(1,0,0,1)+\mu(1,1,1,1)$$

$$\begin{cases} \alpha+\beta+\gamma+\mu = x \\ \beta+\mu = y \\ \alpha+\mu = z \\ \gamma+\mu = t \end{cases} \quad \begin{cases} \gamma = t-\mu \ ; \\ \alpha = z-\mu \ ; \\ \beta = y-\mu \ ; \end{cases} \quad \begin{cases} z-\mu+y-\mu+t-\cancel{\mu}+\cancel{\mu} = x \Rightarrow \mu = \frac{-1}{2}x+\frac{1}{2}y+\frac{1}{2}z+\frac{1}{2}t \\ \gamma = \frac{1}{2}x-\frac{1}{2}y-\frac{1}{2}z+\frac{1}{2}t \\ \alpha = \frac{1}{2}x-\frac{1}{2}y+\frac{1}{2}z-\frac{1}{2}t \end{cases}$$

$$\beta = \frac{1}{2}x+\frac{1}{2}y-\frac{1}{2}z-\frac{1}{2}t$$

Vemos que $\exists\, \alpha,\beta,\gamma,\mu$ tal que en combi lineal con el sist. de vectores generan todo $\bar{x} \in \mathbb{R}^4$.

■ Luego N es base de $\mathbb{R}^4$

(20b)

d) Dim y base del subesp. $G = \{(\beta-\mu,\ 2\gamma+2\mu,\ -\gamma-\mu,\ -\beta+\gamma+2\mu)_N \in \mathbb{R}^4,\ \beta,\gamma,\mu \in \mathbb{R}\} \subset \mathbb{R}^4$

$$G = \beta(1,0,0,-1)_N + \mu(-1,2,-1,2)_N + \gamma(0,2,-1,1)_N \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 2 \\ 0 & 2 & -1 & 1 \end{pmatrix} \xrightarrow{+f_1} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 1 \\ \cancel{0} & \cancel{2} & \cancel{-1} & \cancel{1} \end{pmatrix}$$

$, \beta, \mu, \gamma \in \mathbb{R}$

$\Rightarrow \{(1,0,0,-1)_N, (-1,2,-1,2)_N\}$ es base de G, $\dim_{\mathbb{R}} G = 2$

e)

$$F_{\bar{r}} = \{(x,y,z,t) \in \mathbb{R}^4 \mid \overset{(i)}{ax+by+cz+dt=0}\}, \quad \bar{r} = (a,b,c,d)$$

$$F_{\bar{s}} = \{(x,y,z,t) \in \mathbb{R}^4 \mid \overset{(ii)}{\alpha x+\beta y+\gamma z+\mu t=0}\}, \quad \bar{s} = (\alpha,\beta,\gamma,\mu)$$

$$F_{\bar{r}} \cap F_{\bar{s}} = \{(x,y,z,t) \in \mathbb{R}^4 \mid (i) \wedge (ii)\}, \quad \text{para } \bar{r}, \bar{s}$$

$G = F_{\bar{r}} \cap F_{\bar{s}} \Rightarrow (i) \wedge (ii) =$ implícita de G.

$\forall (x,y,z,t) \in G,\ (x,y,z,t) = \beta(1,0,0,-1) + \mu(-1,2,-1,2) + \gamma(0,2,-1,1)$

$$\begin{cases} x = \beta - \mu \\ y = 2\gamma + 2\mu \\ z = -\gamma - \mu \\ t = -\beta + \gamma + 2\mu \end{cases} \qquad \begin{cases} \beta = x + \mu \\ \mu = \frac{y}{2} - \frac{2t}{2} - \frac{2x}{2} + \frac{2\mu}{2} \Rightarrow * \\ z = -t - x + \cancel{\mu} - \cancel{\mu} \\ \gamma = t + x + \mu - 2\mu = t + x - \mu \end{cases}$$

$* \Rightarrow \cancel{\mu} = \frac{1}{2}y - t - x + \cancel{\mu} \Rightarrow$

$x + t - \frac{1}{2}y = 0$

$G = \{(x,y,z,t) \in \mathbb{R}^4 \mid 2x + 2t - y = 0,\ x + t + z = 0\}$

$$G = F_{\bar{r}} \cap F_{\bar{s}} \Leftrightarrow \left.\begin{array}{l} ax + by + cz + dt = 2x + 2t - y \\ \alpha x + \beta y + \gamma z + \mu t = x + t + z \end{array}\right\} \Rightarrow \begin{array}{l} a = 2, b = -1, d = 2, c = 0 \\ \alpha = 1, \beta = 0, \gamma = 1, \mu = 1 \end{array}$$

Luego $\bar{r} = (2,-1,0,2)$, $\bar{s} = (1,0,1,1)$