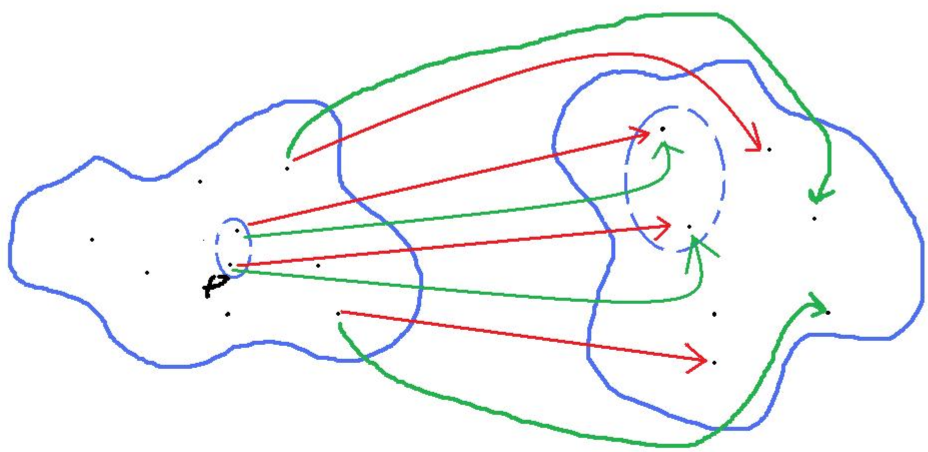
Part IV Tensor Fields

In this section we introduce physical "stuff".

1 Germs

Consider two differentiable manifolds and the set of all mappings from an open neighborhood of a point p in the first manifold to the second. Two such mappings are *equivalent* if we can find an open neighborhood of the point p such that the functions map the same points of that neighborhood to the same points of the target neighborhood.



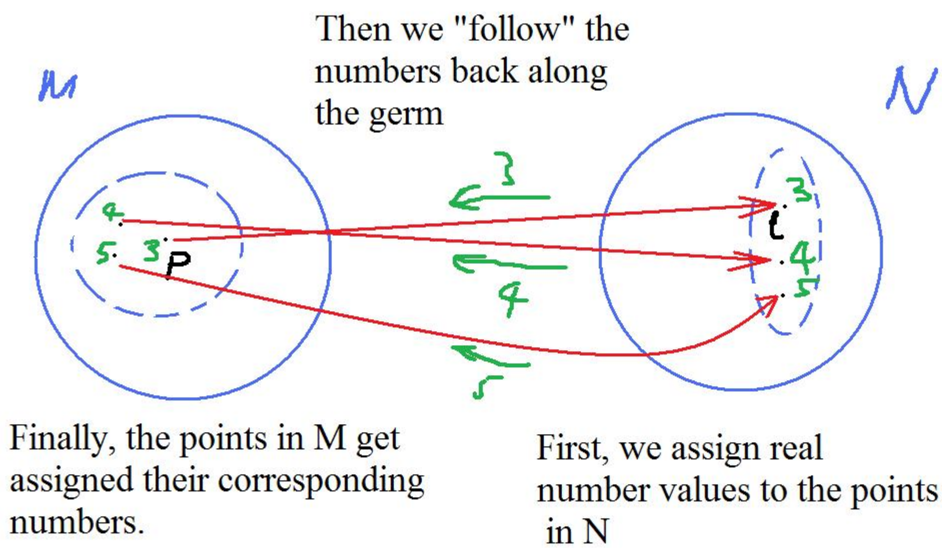
In the above, the red and green functions are equivalent because there exists an open set where they "do the same thing" even if they differ everywhere else. The set of all functions at a point which are equivalent in this way is called a *germ* at that point. The whole idea here is that, if we have a germ at a point, then we can always get "close enough" to the point so that all the functions are equal.

2 Homomorphism of Algebras

Remember that the real line is also a differentiable manifold. We consider a special class of germs which, at any point, assign a single real number to every

point in an open neighborhood of that point. Such a germ is called a *germ of functions* at a point. Germs of functions at points are important because they can represent the value of some physical quantity over a region of spacetime. For example, the density of a material, the masses of particles, temperature, etc.

Now consider two differentiable manifolds M and N, a differentiable germ between them, and the set of all germs of functions around a point q of N. Then, we can define a *homomorphism of algebras* which maps the germs of functions at a point q of N to the germs of functions at a point p of M. We can think of this process as assigning numbers to events in the neighborhood of q and then "pulling back" the numbers to the neighborhood of p along the arrows of the differentiable germ. This process is illustrated below:



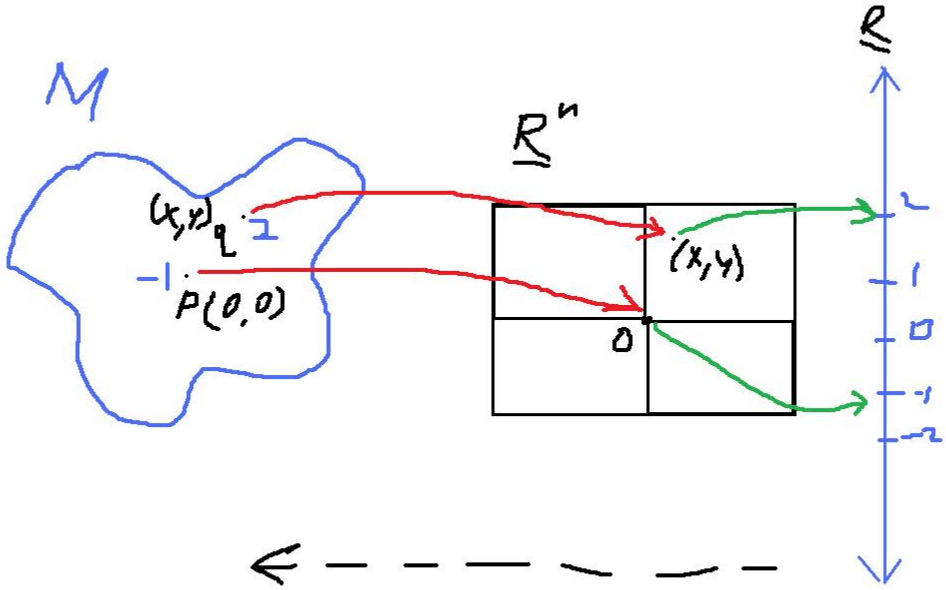
Homomorphisms of algebras are useful because they let us assign physical quantities to complicated situations in spacetime using simpler situations in Euclidean spaces, as we will do in the next section.

3 Charts as Germs

Since charts are mappings from manifolds to a Euclidean space and a Eucliden space is a differentiable manifold, we can use charts to define germs. The germs of a chart at a point all assign the same coordinates to the points in some neighborhood. Additionally, just as we defined germs of functions on arbitrary manifolds, we can do the same with Euclidean space. That is, we can assign a

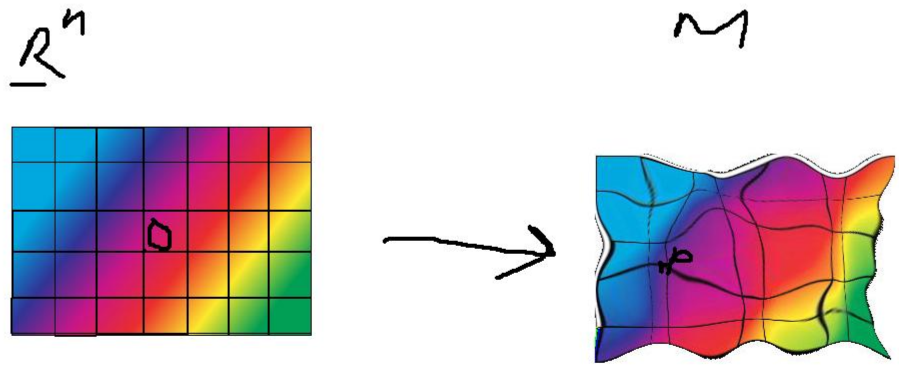
single real number to every point of a Euclidean space using germs of functions.

We use a simple two-step process to assign physical quantities and coordinates to events in spacetime. First, we assign a physical quantity to the points in a Euclidean space using germs of functions. Then, we use germs defined by charts to assign coordinates (and they carry the assigned quantities with them) to events in spacetime, as shown below:



Quantities assigned opposite of mappings.

We can think of this as a layered structure, and an easy way to grasp the whole thing is to squish it all into one object. Below, colors represent real numbers.

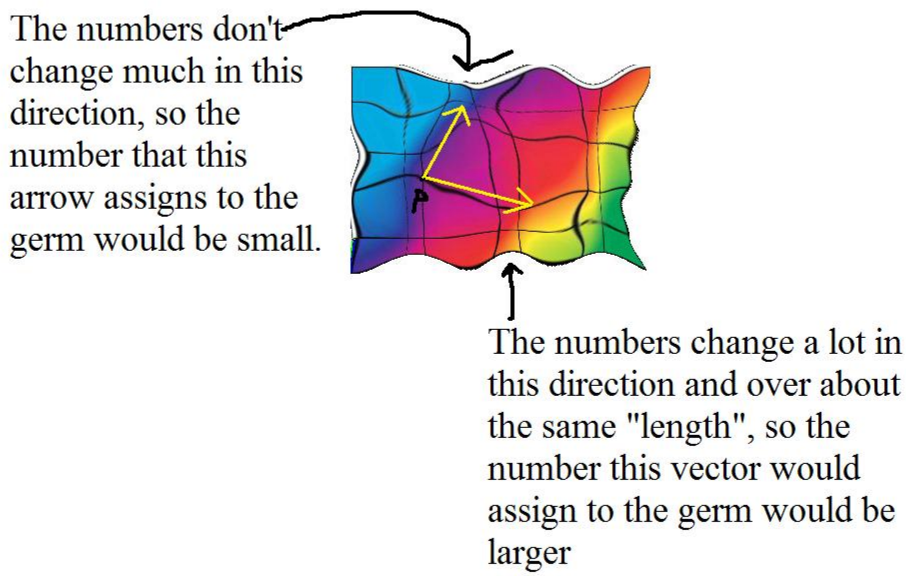


Colors represent real numbers. Every point in the Euclidean space is assigned a real number.

Points in the manifold are assigned both coordinates and real numbers.

4 Tangent Space

Consider a germ of functions defined at a point as in the previous section. Let's draw an arrow at that point. This arrow is called a *tangent vector*. A tangent vector is a function that takes each germ of functions at the point and assigns it a real number such that the real number represents the "rate of change" of the values of the germ in the direction of the arrow.



The set of all tangent vectors at a point is called a *tangent space*. Now, if we go to every point of a manifold and pick out a single tangent vector from each tangent space, then we have a *vector field*.

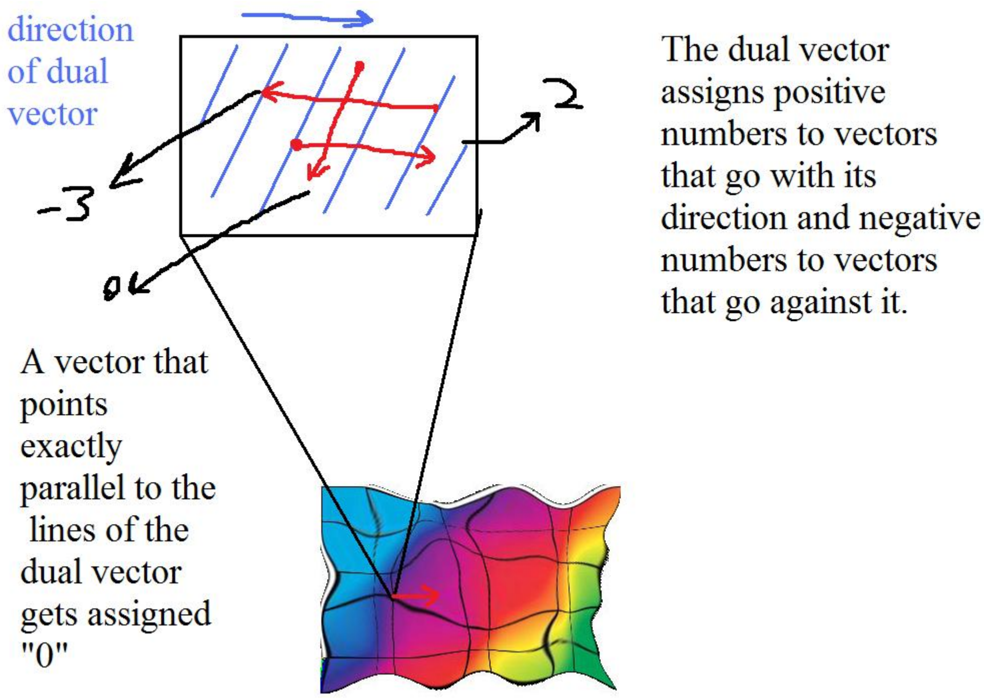
Physically, tangent vectors are used to represent physical quantities that have both magnitude and direction, i.e., velocities, momenta, forces, etc. A vector field, therefore, represents a physical property of a system across all of spacetime. For example, in Newtonian Gravity, gravity is represented as a vector field. Each point of space is assigned a magnitude and direction (the force and direction of gravitational pull) that a particle at that point would experience

5 Dual Vectors

Here, we defined vectors using scalar quantities. In practice, however, it turns out that we often know the vectors associated with a physical system and we have to find a scalar quantity. For example, we might be given all the forces acting on a system (a vector field) and our job might be to find the distance traveled by its particles (a scalar quantity).

For this job, we have what are called “dual vectors”. A *dual vector* at a point is a function which takes each vector and assigns it a real number in a linear way. We can visualize dual vectors as “topographic maps”. Where the lines (the technical term is “hypersurfaces”, since in 3d the topographic “lines” would have to be topographic “sheets”) of the dual vectors are close together, that means that the germ of functions is changing quickly with respect to a coordinate line.

Where they are far apart, it is changing slowly. Just like vectors, dual vectors have both a sense of direction and a magnitude. So if we have a tangent vector at a point as well as a dual vector, the scalar that the dual vector assigns to the tangent vector is equal to the number of hypersurfaces that the vector punches through.



Just like vectors, dual vectors have a "dual space". The set of all dual vectors at a point is called a *dual space* at the point. Also, we have *dual vector fields*, which is a choice of dual vector from each dual space of each event of a manifold.

Dual vectors are mathematically equivalent to tangent vectors and can be used to represent directed physical quantities just as well. Since they are equivalent, we can interchange their roles. That is, just as dual vectors are like machines that input vectors and output real numbers, vectors can be thought of as machines that input dual vectors and output real numbers. Thus, once we know the geometry of a physical system (i.e. its representation in terms of vectors and dual vectors) we can always predict the scalar quantities that we want to measure.

6 Tensor Fields

However, a geometric description of real physical system requires the use of geometric objects which are a bit more complicated than vectors and dual vectors. These geometric objects are called "tensors". A tensor at a point can be thought of as a machine that inputs some number of vectors and some number

of dual vectors and outputs a real number. The rank of a tensor is given by two numbers, (r, s). The first number is the number of vectors that the tensor needs as input, and the second number is the number of dual vectors that the tensor needs as input. A rank (0, 0) tensor does not need any vectors or dual vectors as input. A rank (0, 0) tensor at a point is simply a real number. A rank (1, 0) tensor needs only a vector as input to produce a scalar. Since dual vectors are the objects which need a single vector as input to output a scalar, rank (1, 0) tensors are dual vectors. Similarly, regular vectors are rank (0, 1) tensors. We could go on and on with rank (1, 1) tensors, (2, 0) tensors, (5, 6) tensors, etc.

A *tensor field* is a choice of tensor at each point on a manifold. It is tensor fields that represent pretty much everything that exists in the universe (including you). General Relativity deals with three main kinds of tensor fields. The first is the stress-energy tensor. It describes pretty much any form of matter or radition in the universe. It is a rank (2, 0) tensor, so you need to feed it two vectors in order to get a scalar quantity from it. It is also the soruce of the gravitational field in GR (as we will later see). The second tensor field is called the electromagnetic tensor field. It is a rank (2, 0) tensor and it describes the electromagnetic properties of objects. The final kind of tensor field is the most important in General Relativity. It is called the *metric tensor*. The metric tensor is a rank (2, 0) tensor (notice a pattern?) and it has the following properties:

1. $g(X, Y) = g(Y, X)$, for all vectors X and Y.
2. $g(X, Y) = 0$ if and only if $Y = 0$.

The metric tensor describes the length and angle relationship of vectors at each point. A rough interpretation of the metric tensor is that it gives you the length of an infinitely small piece of a curve at a point along the direction of X as it bends in the direction of Y. The first property says that it doesn't matter which role each vector takes. We could just as well say that the metric tensor gives the length of an infinitely small piece of a curve at a point along the direction of Y as it bends in the direction of X. The second property says that, if you don't measure in any direction at all, or if the curve doesn't go in any direction, then the lenth of an infinitely small piece of it is 0.

Tensor describe the universe in a way that does not depend on the coordinate system used. The way we defined vectors, dual vectors, and then tensors, we only used the germs of functions on manifolds (which represent measurable properties of things) and never referred to a specific coordinate system.

7 Conclusion

We have all the mathematical concepts we need to define spacetime. The next part will be more philosophically oriented and will investigate the reasons and arguments for why a coordinate-independent description of the universe was necessary.