أفكار التكامل الغير محدود

① Direct :-

* عن طريق تعويضة عادية ، بوصل التكامل لصورة مباشرة .

Ⓘ $I_1 = \int \frac{\sin 2x}{\sqrt{1+\sin x}}\, dx$ let $\sin x = u$

$\cos x\, dx = du$.

$= 2\int \frac{u}{\sqrt{1+u}}\, du.$ لما أشوف الشكل ده لازم أجمع واحد وأطرحه

لأخليها زي الفوق

$= 2\int \frac{u+1}{\sqrt{u+1}}\, du - 2\int \frac{1}{\sqrt{u+1}}\, du.$

$= \frac{2*2(u+1)^{3/2}}{3} - 2*2\sqrt{u+1} + c. = \frac{4}{3}(\sin x+1)^{3/2} - 4\sqrt{\sin x+1}$

Ⓘⓘ $I_2 = \int (x-1)^5 (x+1)^2\, dx$ let $x-1 = u$

$dx = du$.

لما أشوف حاجة مرفوعة لـ power كبير بسميها u.

$= \int u^5 (u+2)^2\, du. = \int u^5 (u^2+4u+4)\, du$

$= \int u^7\, du + 4\int u^6\, du + 4\int u^5\, du.$

$= \frac{u^8}{8} + 4\frac{u^7}{7} + 4\frac{u^6}{6} + c = \frac{(x-1)^8}{8} + \frac{4}{7}(x-1)^7 + \frac{2}{3}(x-1)^6$

(III) $I_3 = \int \frac{\cos x + \sin 2x}{\sqrt{1+\sin^2 x}}\,dx$

لما أشوف له جة زي دي، والي فوق مش تفاضل اللي تحت الجذر، بفكها

$$= \int \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx + \int \frac{\sin 2x}{\sqrt{1+\sin^2 x}}\,dx$$

$d(\sin x) = \cos x\,dx$

$d(\sin^2 x) = 2\sin x \cos x\,dx = \sin 2x\,dx$

$$= \sinh^{-1}(\sin x) + 2\sqrt{1+\sin^2 x}\,dx$$

(IV) $I_4 = \int \frac{\tanh x}{\sqrt{3\sinh^2 x + 9}}\,dx$

لما أشوف له جة زي دي، بفك tanh، وأخليها كلها sinh, cosh

$$= \int \frac{\sinh x}{\cosh x\sqrt{3\sinh^2 x + 9}}\,dx$$

ببص على التفاضل الموجود عندي

$d(\cosh x) = \sinh x\,dx$ ، ∴ هعوض $u = \cosh x$

$$= \int \frac{du}{u\sqrt{3(u^2-1)+9}} = \frac{1}{\sqrt{3}}\int \frac{du}{u\sqrt{u^2+2}} = \frac{1}{\sqrt{3}\cdot\sqrt{2}}\operatorname{csch}^{-1}\frac{u}{\sqrt{2}} + c$$

$$\therefore I = \frac{1}{\sqrt{6}}\operatorname{csch}^{-1}\left(\frac{\cosh x}{\sqrt{2}}\right) + c.$$

② إكمال المربع.

Ⓘ $I_5 = \int \frac{x}{\sqrt{x^2-2x}}\,dx$

* طالما البسط أقل منه المقام درجة واحدة، لازم أخلي البسط تفاضل المقام. ؛ بعديه إكمال مربع للجزء الثاني

$$= \frac{1}{2}\int \frac{(2x-2)+2}{\sqrt{x^2-2x}}\,dx = \frac{1}{2}\int \frac{2x-2}{\sqrt{x^2-2x}}\,dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2-2x}}\,dx$$

$$= \frac{1}{2}\left(2\sqrt{x^2-2x}\right) + \int \frac{dx}{\sqrt{(x-1)^2-1}} = \sqrt{x^2-2x} + \cosh^{-1}(x-1) + C.$$

Ⓘⓘ $I_6 = \int \frac{\sqrt{x-2}}{\sqrt{5-x}}\,dx \times \frac{\sqrt{x-2}}{\sqrt{x-2}}$ لما يبقى عندي جذر فوق وتحت، بضرب في جذر اللي فوق

$$= \int \frac{x-2}{\sqrt{-x^2+7x-10}} = \frac{-1}{2}\int \frac{-2x+7-7-4}{\sqrt{-x^2+7x-10}}\,dx = \frac{-1}{2}\left[\int \left(\frac{-2x+7}{\sqrt{-x^2+7x-10}} - \frac{11}{\sqrt{-x^2+7x-10}}\right)dx\right]$$

$$= \frac{-1}{2}\left(2\sqrt{-x^2+7x-10}\right) + \frac{11}{2}\int \frac{dx}{\sqrt{-\left[(x-\frac{7}{2})^2 - \frac{49}{4} + 10\right]}}$$

$$= -\sqrt{-x^2+7x-10} + \frac{11}{2}\left(\sin^{-1}\frac{x-7/2}{\sqrt{9/4}}\right) = -\sqrt{-x^2+7x-10} + \frac{11}{2}\sin^{-1}\frac{(x-7/2)2}{3} + C$$

III $\frac{I}{7} = \int \frac{\cosh x}{\sqrt{3(\cosh^2 x) - \sinh x + 1}}\, dx$

let $u = \sinh x$

$du = \cosh x\, dx$

طالما عندي $d(\sinh x) = \cosh x\, dx$ نعوض

فنعوض $(\sinh x)$ ب u

$$= \int \frac{du}{\sqrt{3(1+u^2) - u + 1}} = \frac{1}{\sqrt{3}} \int \frac{du}{\sqrt{u^2 - \frac{1}{3}u + \frac{4}{3}}}$$

$$= \frac{1}{\sqrt{3}} \int \frac{du}{\sqrt{(u - \frac{1}{6})^2 - \frac{1}{36} + \frac{4}{3}}} = \frac{1}{\sqrt{3}} \int \frac{du}{\sqrt{(u - \frac{1}{6})^2 + \frac{47}{36}}}$$

$$= \frac{1}{\sqrt{3}} \sinh^{-1}\left(\frac{\sinh x - \frac{1}{6}}{\sqrt{\frac{47}{36}}} \right) + C.$$

③ الكسور الجزئية

أولاً لما نلاقي في المقام معادلة سهلة التحليل .

① $\frac{I}{8} = \int \frac{x+3}{x^3+x^2+2x+2}\,dx$ -

$$= \int \frac{(x+3)\,dx}{x^2(x+1)+2(x+1)} = \int \frac{x+3}{(x+1)(x^2+2)}\,dx$$

$$= \int \left(\frac{A}{x+1} + \frac{Bx+C}{x^2+2}\right) dx \quad , \quad A = \frac{2}{3}$$

$$(x^2+2)A + (x+1)(Bx+C) = x+3$$

بمقارنة الحدود

$$Bx^2 + Ax^2 = 0$$

$$A = 2/3 \qquad \therefore B = -2/3$$

$$C + 2A = 3 \qquad A = 2/3 \qquad \therefore C = 5/3$$

$$\therefore I = \int \left(\frac{2/3}{x+1}\right) dx + \int \left(\frac{-2/3\,x + 5/3}{x^2+2}\right) dx \rightarrow \frac{-1}{3}\int \frac{2x}{x^2+2}\,dx + \frac{5}{3}\int \frac{dx}{x^2+}$$

$$= \frac{2}{3}\ln|x+1| - \frac{1}{3}\ln|x^2+2| + \frac{5}{3}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c.$$

(11) $I_9 = \int \frac{x^4+3x^2-1}{x^4-1}\,dx$

$$= \int \frac{x^4-1+3x^2}{x^4-1}\,dx = \int 1\,dx + \int \frac{3x^2}{x^4-1}\,dx$$

$$= x + 3\int \frac{x^2}{(x^2+1)(x+1)(x-1)}\,dx = x + 3\left[\int \left(\frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}\right)dx\right]$$

$$A = \frac{1}{(1+1)(1+1)} = \frac{1}{4}$$

$$B = \frac{1}{(1+1)(-2)} = \frac{-1}{4}$$

$$x^2 = A(x^2+1)(x+1) + B(x^2+1)(x-1) + \overbrace{(x-1)(x+1)}^{(x^2-1)}(Cx+D)$$

$$0x^3 = Ax^3 + Bx^3 + Cx^3$$

$$\therefore C = 0$$

$$x^2 = Ax^2 \cancel{+} - Bx^2 + Dx^2$$

$$1 = \frac{1}{2} + D \qquad \therefore D = \frac{1}{2}$$

$$\therefore I = x + 3\int\left(\frac{1/4}{x-1} + \frac{(-1/4)}{x+1} + \frac{1/2}{x^2+1}\right)dx$$

$$= x + \frac{3}{4}\ln|x-1| - \frac{3}{4}\ln|x+1| + \frac{3}{2}\tan^{-1}x + c.$$

let $\sinh x = u$

$\cosh x\,dx = du$

(III) $I_6 = \int \frac{\cosh x}{(\sinh^2 x - 5\sinh x + 6)} dx$

$= \int \frac{du}{u(u-2)(u-3)} = \int \left(\frac{A}{u} + \frac{B}{u-2} + \frac{C}{u-3}\right) du.$

$A = \frac{1}{(-2)(-3)} = \frac{1}{6}$, $B = \frac{1}{2(2-3)} = -1/2$, $C = \frac{1}{3(3-2)} = 1/3$

$\therefore I = A \ln|u| + B \ln|u-2| + C \ln|u-3| + Com$

$= \frac{1}{6} \ln|\sinh x| - \frac{1}{2} \ln|\sinh x - 2| + \frac{1}{3} \ln|\sinh x - 3| + C.$

④ تكاملات قوى الدوال المثلثية:

~~I_{11}~~

(I) $I_{11} = \int \sin^5 x \sqrt{\cos x}\, dx$

نجيب واحدة من $\sin^4 x$

$= -\int (1-\cos^2 x)^2 \sqrt{\cos x} \cdot (-\sin x) dx$

$= -\int (1 + \cos^4 x - 2\cos^2 x)\, d(\cos x) = \int -d\cos x - \int \cos^4 x \cdot d\cos x + 2\int \cos^2 x \cdot d\cos x$

$= -\cos x - \frac{\cos^5 x}{5} + \frac{2}{3} \cos^3 x + c.$

HINTS:-

$\sin^m, \cos^n$

① if Both even ⟹ Double angle Rules :

② if one ODD, on even ⟹ we take one of it to make it even

③ if Both are odd, ⟹ we take one of the small power and make it even.

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

~~$\cos(a+b)x$~~

$\cos ax \cdot \cos bx = \frac{1}{2}(\cos(a+b)x + \cos(a-b)x)$

$\sin ax \sin bx = \frac{1}{2}(\cos(a-b) - \cos(a+b))$

(II) $I_{12} = \int \frac{\tan^3(\tan x) \cdot \sec^4(\tan x)}{\cos^2 x} dx$ let $\tan x = u$

$\sec^2 x\, dx = du$

$= \int \tan^3(u) \cdot \sec^4(u)\, du.$

$= \int \tan^2(u) \cdot \sec^3(u) \cdot \sec(u) \tan(u)\, du = \int \tan^2(u) \cdot \sec^3(u)\, d\sec(u)$

$= \int (\sec^2(u) - 1) \sec^3(u)\, d\sec(u)$

$= \int \sec^5(u) \cdot d\sec(u) - \int \sec^3(u)\, d\sec(u) = \frac{\sec^6(\tan x)}{6} - \frac{\sec^4(\tan x)}{4} + C.$

(III) $I_{13} = \int \cot^5\left(\frac{\theta}{6}\right) \sec^2\left(\frac{\theta}{6}\right) d\theta$ let $\frac{\theta}{6} = x$

$d\theta = 6\, dx$

$= 6 \int \frac{\cos^5 x}{\sin^5 x} \cdot \cos^2 x\, dx = 6 \int \cos^7 x \cdot \sin^{-5} x\, dx$

$\cancel{6\int (1-\cos}$ $= 6 \int (1 - \sin^2 x)^3 \cdot \sin^{-5} x \cdot d\sin x$

$= 6 \int (\sin^{-5} x + 3 \sin^{-1} x - 3 \sin^{-3} x - \sin x)\, d\sin x$

$= -\frac{6}{4} \sin^{-4} \frac{\theta}{6} + 18 \ln|\sin x| + \frac{18}{2} \sin^{-2} x - 6 \frac{\sin^2 x}{2} + C$

$= \frac{-3}{2} \sin^{-4} \frac{\theta}{6} + 18 \ln\left|\sin \frac{\theta}{6}\right| + 9 \sin^{-2} \frac{\theta}{6} - 3 \sin^2 \frac{\theta}{6} + C.$

HINTS:-

$\tan^m$, $\sec^n$ (powers)

if Both are even $\Longrightarrow$ we take $\sec^2$

if Both are odd $\Longrightarrow$ we take sec tan

$1 - \tanh^2 x = \operatorname{sech}^2 x$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

$\coth^2 x - 1 = \operatorname{csch}^2 x$

if the Power of tan is odd, and the Power of sec is even $\Longrightarrow$

- if the Power of tan is larger than the Power of sec $\Longrightarrow$ take $\sec^2$
- if the Power of sec is larger than the Power of tan $\Longrightarrow$ take sec tan

أ/ الجبري

(I) $I_{14} = \int \overset{u}{(\sinh^{-1}x)^2}\, \overset{dv}{dx}$

$= x(\sinh^{-1}x)^2 - \int x * 2\sinh^{-1}x * \frac{dx}{\sqrt{1+x^2}}$

$= x(\sinh^{-1}x)^2 - [\sinh^{-1}x * 2\sqrt{1+x^2} - 2\int \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\,dx]$

$= x(\sinh^{-1}x)^2 - 2\sqrt{1+x^2}\,\sinh^{-1}x - 2x + c.$

(II) $I_{15} = \int x\sqrt{\cosh x+1}\,dx$

$= \int x\sqrt{2\cosh^2\frac{x}{2}}\,dx = \sqrt{2}\int \overset{u}{x} . \overset{dv}{\cosh\frac{x}{2}\,dx}$

$I = \sqrt{2}[x . 2\sinh\frac{x}{2} \cancel{d} - 2\int \sinh\frac{x}{2}\,dx$

$= 2\sqrt{2}\,x\sinh\frac{x}{2} - 4\sqrt{2}\cosh\frac{x}{2} + c.$

$$\boxed{\begin{aligned} &\cos x+1 = 2\cos^2\frac{x}{2}\\ &\cos x+1 = 2\sin^2\frac{x}{2}\\ &\cosh x+1 = 2\cosh^2\frac{x}{2}\\ &\cosh x-1 = 2\sinh^2\frac{x}{2}\end{aligned}}$$

(III) $I_{16} = \int \frac{x^3}{\sqrt[3]{x^2+4}}\,dx =$

$= \frac{1}{2} \int \frac{2x}{\sqrt[3]{x^2+4}}\, x^2\,dx \cancel{= \frac{1}{2}\left[x^2 \cdot \frac{(x^2+4)^{2/3}}{4}\right] + \frac{2}{4}\int}$

$= \frac{1}{2}\left[x^2 * \frac{3(x^2+4)^{2/3}}{2} - \frac{3}{2}\int (x^2+4)^{2/3} * 2x\,dx\right.$

$= \frac{3}{4} x^2 (x^2+4)^{2/3} - \frac{9}{10}(x^2+4)^{5/3} + C.$

(6) التعويضات المثلثية والزائدية لإزالة الجذر التربيعي

(I) $I_{17} = \int \frac{x^5}{(4-x^2)^{3/2}}\,dx$

$= \int \frac{x^5}{(4-x^2)\sqrt{4-x^2}}\,dx$ let $x = 2\sin\theta$

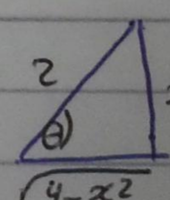
$dx = 2\cos\theta\,d\theta$

$= 2^5 \int \frac{\sin^5\theta\,d\theta \cdot \cos\theta}{2 * 4(1-\sin^2\theta)\sqrt{1-\sin^2\theta}} = 4 \int \sin^5\theta \cdot \cos^{-3}\theta\,d\theta$

$= -4 \int (1-\cos^2\theta)^2 \cdot \cos^{-2}\theta \cdot d(\cos\theta) = -4\int [\cos^{-2}\theta - 2 + \cos^2\theta]\,d(\cos\theta)$

$\cancel{= -4}$

$= -4\frac{\cos^{-1}\theta}{-1} + 8\cos\theta - 4\frac{\cos^3\theta}{3} + C$



$= \frac{4}{\cos\theta} + 8\cos\theta - \frac{4}{3}\cos^3\theta + C$

$= 8\sqrt{4-x^2} + 4\sqrt{4-x^2} - \frac{1}{6}(4-x^2)^{3/2} + C.$

(III) $I_{18} = \int \cosh x \sqrt{2\sinh x - \sinh^2 x}\, dx$ let $\sinh x = u$

$\cosh x\, dx = du$

$= \int \sqrt{-(\sinh^2 x - 2\sinh x)} \cdot \cosh x\, dx = \int \sqrt{-(u^2 - 2u)}\, du.$

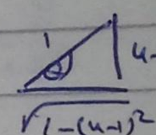
$= \int \sqrt{-((u-1)^2 - 1)} = \int \sqrt{1-(u-1)^2}\, du.$

let $u - 1 = \sin\theta$

$du = \cos\theta\, d\theta$

$= \int \sqrt{1 - \sin^2\theta} \cdot \cos\theta\, d\theta = \int \cos^2\theta\, d\theta$

$= \int \frac{1}{2}\, d\theta + \int \frac{1}{2} \cos 2\theta\, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C.$



$\theta = \sin^{-1}(u-1)$, $\sin 2\theta = 2\sin\theta\cos\theta = 2(u-1)\sqrt{1-(u-1)^2}$

$I = \frac{1}{2}\left[\sin^{-1}(u-1) + (u-1)\sqrt{1-(u-1)^2}\right] + C.$

$= \frac{1}{2}\left[\sin^{-1}(\sinh x - 1) + (\sinh x - 1)\sqrt{1-(\sinh x - 1)^2}\right] + C.$

7 التعويض بنصف الزاوية.

لما يكون عندي دوال مثلثية مجموعة على بعض في صورة بسط ومقام.

① $I_{19} = \int \frac{\sin x}{3+5\sin x}\,dx$

$= \int \frac{2z/1+z^2}{\frac{3+3z^2+10z}{1+z^2}} * \frac{2dz}{1+z^2}$

let $z = \tan\frac{x}{2}$

$dx = \frac{2dz}{1+z^2}$

$\sin x = \frac{2z}{1+z^2}$

$\cos x = \frac{1-z^2}{1+z^2}$

$\tan x = \frac{2z}{1-z^2}$

$= 4\int \frac{z\,dz}{(1+z^2)(3z+1)(z+3)}$

بالكسور الجزئية:

$= 4\int \left[\frac{A}{3z+1} + \frac{B}{z+3} + \frac{Cz+D}{1+z^2}\right] dz$

$A = \frac{-1/3}{80/27} = \frac{-9}{80}$

$B = \frac{-3}{(1+9)(1-9)} = \frac{-3}{80}$

$z = (z+3)(1+z^2)A + (3z+1)(1+z^2)B + (z+3)(3z+1)(Cz+D)$

$0z^3 = Az^3 + 3Bz^3 + 3Cz^3$ بمقارنة المعاملات.

$0 = \frac{-9}{80} - \frac{3}{80} + 3C$ $\therefore C = 4$

الحد المطلق

$0 = 3A + B + 3D$ $\therefore D = \frac{-3A-B}{3} = \frac{-3}{8\times 3} = \frac{-1}{8}$

$\therefore I = 4\left[\frac{1}{3}A\ln|z+1| + B\ln|z+3| + 2\ln|1+z^2| + D\tan^{-1}z\right] + c.$

$= \frac{-3}{20}\ln\left|\tan\frac{x}{2}+1\right| - \frac{3}{20}\ln\left|\tan\frac{x}{2}+3\right| + 8\ln\left|1+\tan^2\frac{x}{2}\right| - \frac{1}{4}x + c.$

$$Ⓘ \quad I_{20} = \int \frac{dx}{4 + \sinh x + \cosh x}$$

let $\tanh \frac{x}{2} = z$

$$dx = \frac{2\,dz}{1 - z^2}$$

$$= \int \frac{2dz/1-z^2}{\frac{4 - 4z^2 + 2z + 1 + z^2}{1 - z^2}}$$

$$\sinh x = \frac{2z}{1 - z^2} \qquad \cosh x = \frac{1 + z^2}{1 - z^2}$$

$$= \int \frac{2\,dz}{-3z^2 + 2z + 5} = \frac{-2}{3} \int \frac{dz}{z^2 + \frac{2}{3} z - \frac{5}{3}}$$

[illegible]

$$I = \frac{-2}{3} \int \frac{dz}{(z + \frac{1}{3})^2 - \frac{1}{9} - \frac{5}{3}}$$

$$= \frac{2}{3} \int \frac{dz}{\frac{16}{9} - (z + \frac{1}{3})^2} = \frac{2}{3} \tanh^{-1} \frac{(z + \frac{1}{3})}{\sqrt{\frac{16}{9}}} * \frac{1}{\sqrt{\frac{16}{9}}} + c,$$

$$= \frac{1}{2} \tanh^{-1} \frac{3(\tanh \frac{x}{2} + \frac{1}{3})}{4} + c,$$

⑧ الاجتزال المتتالي

(I) $I_{21} = \int x \cos^n x \, dx.$

$$= \int \underbrace{x \cdot \cos^{n-1} x}_{u} \cdot \underbrace{\cos x \, dx}_{dv}$$

$$I_n = x \cdot \cos^{n-1} x \cdot \sin x - \int \sin x \left[\cos^{n-1} x \, dx + (n-1) \cos^{n-2} x \cdot (-\sin x) dx\right]$$

$$= x \cdot \cos^{n-1} x \cdot \sin x - \int \cos^{n-1} x \cdot \sin x \, dx + (n-1)(1-\cos^2 x) \cos^{n-2} x \, dx]$$

$$I_n = [x \cos^{n-1} x \cdot \sin x + \frac{\cos^n x}{n} + I_{n-2} (n-1)] \frac{1}{n}$$

(II) $I_{22} = \int e^x \sinh^n x \, dx$

$$= \int \underbrace{e^x \cdot \sinh^{n-1} x}_{u} \cdot \underbrace{\sinh x \, dx}_{dv} = \int$$

$$I_n = e^x \sinh^{n-1} x \cdot \cosh x - \int \cosh x [e^x \sinh^{n-1} x \, dx + (n-1) \sinh^{n-2} x \cdot \cosh x \, dx]$$

$$= e^x \sinh^{n-1} x \cdot \cosh x - \int e^x \sinh^{n-1} x \cdot \cosh x \, dx - (n-1) e^x \sinh^{n-2} x (1+\sinh^2 x) dx$$

$$I_n = [e^x \sinh^{n-1} x \cdot \cosh x - (n-1) I_{n-2} - \int I'] \frac{1}{n}$$

$$I' = \int \underbrace{e^x}_{u} \cdot \underbrace{\sinh^{n-1} x \cdot \cosh x \, dx}_{dv} = e^x \frac{\sinh^n x}{n} - \frac{1}{n} \int \underbrace{\sinh^n x \, e^x \, dx}_{I_n}$$

$$\therefore I_n = \frac{n}{n^2-1} [e^x \sinh^{n-1} x \cdot \cosh x - (n-1) I_{n-2} - \frac{e^x \sinh^n x}{n}] + C.$$